

LOCAL GEOMETRY EFFECT ON THE SOLUTIONS OF EVOLUTION EQUATIONS: THE CASE OF THE SWIFT-HOHENBERG EQUATION ON CLOSED MANIFOLDS

NIKOLAOS ROIDOS

ABSTRACT. We consider the Swift-Hohenberg equation on manifolds with conical singularities and show existence, uniqueness and maximal L^q -regularity of the short time solution in terms of Mellin-Sobolev spaces. We also provide information about the asymptotic behavior of the solution near the singularity. Furthermore, we give a necessary and sufficient condition such that the above solution exists for all times. Then, we consider the same problem on closed manifolds and obtain similar results by employing the above singular analysis theory. In this case, we further provide a relation between the local geometry around any point and the dependence of the solution on the geodesic distance to that point. The above approach can be applied to different problems.

1. INTRODUCTION

The main task of this article is to demonstrate a method for obtaining local expansion in space variables of the solution of an evolution equation on a closed manifold by employing a singular analysis approach. Although the concept is applicable to several problems, in order to be more precise we focus on a *prototype* semilinear one.

The Swift-Hohenberg equation. *Consider the problem*

$$(1.1) \quad u'(t) + (\Delta + 1)^2 u(t) = F(t, u(t)), \quad t \in (0, T),$$

$$(1.2) \quad u(0) = u_0,$$

where $T > 0$ is finite and $F(s, x)$ is a polynomial over x with s -dependent coefficients that are Lipschitz continuous on \mathbb{R} .

The above problem, known as Swift-Hohenberg equation, has applications to many physical domains and can model e.g. thermally convecting fluid flows [28], cellular flows [21], phenomena in optical physics [29] etc. It is also well known for its pattern formation under evolution (see e.g. [5], [18], [19] and [30]). Here u is regarded as a scalar field.

We treat the above problem on a closed manifold and show existence, uniqueness and maximal L^q -regularity of the short time solution. Furthermore, we provide certain condition that is necessary and sufficient for the above solution to exist for all times. In order to do this, instead of using standard ellipticity combined with e.g. maximal L^q -regularity theory for the Laplacian on the usual Sobolev spaces, we choose to work on geodesic polar coordinates and to regard the problem as in the case of a manifold with *warped conical singularities*. Thus, the problem becomes equivalent to a fourth order degenerate parabolic semilinear partial differential equation (PDE), which we solve by using singular analysis theory.

The main advantage of the above approach is that we can use the fact that well-posed solutions of conically degenerate PDEs have precise asymptotic behavior close to the singularity that is determined by the geometry of the cross section of the cone. Therefore, we can transfer this knowledge to the non-singular case of a closed manifold and provide a relation between the local geometry around a point and the local asymptotic expansion of the solution of the equation in terms of the geodesic distance to that point. It turns out that e.g. on areas around points with sufficiently large local curvature we have strong radial (with respect to the geodesic distance to those points) inhomogeneity of the solution.

Date: December 30, 2016.

2010 Mathematics Subject Classification. 35K65; 35K90; 35K91.

The theory we use to solve the degenerate equation is maximal L^q -regularity theory for linear and quasilinear parabolic problems. The short time solution is obtained by a theorem of Clément and Li. Furthermore, we proceed to the existence of a long time solution having maximal L^q -regularity by imposing an additional assumption to the right hand side of (1.1), which turns out to be necessary for this property. Namely, we show that if the L^q -norm of the term $F(t, u(t))$ is bounded by a positive continuous function of T , then the Banach fixed point argument of the Clément and Li theorem can be successively repeated and hence the well-posed solution exists for all times. The method we use to show this is based on interpolation space estimates obtained by using a maximal L^q -regularity inequality for linear parabolic problems. The above method can be generalized to the fully quasilinear case and provide similar results.

Concerning the problem (1.1)-(1.2) on a manifold with conical singularities, we regard the Laplacian as a *cone differential operator* acting on *Mellin-Sobolev spaces* and develop the related maximal L^q -regularity theory. More precisely, among all possible closed extensions of the Laplacian, we choose one that satisfies the property of maximal L^q -regularity, and whose domain description turns out to be related to the geometry of the cross section of the cone. We further collect the necessary machinery related to the theory of cone differential operators on Mellin-Sobolev spaces with emphasis on the non-linear PDEs point of view.

2. MAXIMAL L^q -REGULARITY THEORY FOR LINEAR AND QUASILINEAR PARABOLIC PROBLEMS

In this section we present some basic functional analytic machinery related to the property of maximal L^q -regularity for linear and quasilinear parabolic problems. We start with the elementary property of sectoriality that guarantees existence and uniqueness of the *weak* solution of the linearized problems. Let $X_1 \xrightarrow{d} X_0$ be a continuously and densely injected complex Banach couple.

Definition 2.1 (Sectorial operator). *For $\theta \in [0, \pi)$ denote by $\mathcal{P}(\theta)$ the class of all closed densely defined linear operators A in X_0 such that*

$$S_\theta = \{\lambda \in \mathbb{C} \mid |\arg \lambda| \leq \theta\} \cup \{0\} \subset \rho(-A) \quad \text{and} \quad (1 + |\lambda|)\|(A + \lambda)^{-1}\|_{\mathcal{L}(X_0)} \leq K_{A,\theta}, \quad \lambda \in S_\theta,$$

for some $K_{A,\theta} \geq 1$ that is called sectorial bound of A and depends on A and θ . The elements in $\mathcal{P}(\theta)$ are called sectorial operators (of positive type) of angle θ .

If for an operator the sectoriality property holds in some domain, then it can be extended to a bigger domain by possibly increasing the sectorial bound, as the following elementary result guarantees (see e.g. the Appendix of [24] or equivalently [1, III.4.7.11]).

Lemma 2.2. *Let $\Omega \subset \mathbb{C}$ be any set and A be a closed linear densely defined operator in a complex Banach space X_0 such that $\Omega \subset \rho(-A)$ and $(1 + |\lambda|)\|(A + \lambda)^{-1}\|_{\mathcal{L}(X_0)} \leq K_{A,\Omega}$, for any $\lambda \in \Omega$ and some $K_{A,\Omega} \geq 1$. Then,*

$$\Omega' = \bigcup_{\lambda \in \Omega} \left\{ \mu \in \mathbb{C} \mid |\mu - \lambda| \leq \frac{1 + |\lambda|}{2K_{A,\Omega}} \right\} \subset \rho(-A)$$

and

$$\|(A + \mu)^{-1}\|_{\mathcal{L}(X_0)} \leq \frac{2K_{A,\Omega} + 1}{1 + |\mu|} \quad \text{for all } \mu \in \Omega'.$$

Proof. By the relation

$$A + \mu = (A + \lambda)(I + (\mu - \lambda)(A + \lambda)^{-1}),$$

we have that $\Omega' \subset \rho(-A)$ and for any $\mu \in \Omega'$ the following estimate holds

$$\begin{aligned} & \|(A + \mu)^{-1}\|_{\mathcal{L}(X_0)} \\ & \leq \|(I + (\mu - \lambda)(A + \lambda)^{-1})^{-1}\|_{\mathcal{L}(X_0)} \|(A + \lambda)^{-1}\|_{\mathcal{L}(X_0)} \leq \frac{2K_{A,\Omega}}{1 + |\lambda|} \\ & \leq \frac{2K_{A,\Omega}(1 + |\lambda| + |\mu - \lambda|)}{(1 + |\lambda|)(1 + |\mu|)} \leq \frac{2K_{A,\Omega}}{(1 + |\mu|)} \left(1 + \frac{1}{2K_{A,\Omega}}\right) = \frac{2K_{A,\Omega} + 1}{1 + |\mu|}. \end{aligned}$$

□

Sectorial operators define holomorphic functional calculus by the use of Dunford integral formula. For any $\rho \geq 0$ and $\theta \in (0, \pi)$, let the positively oriented path

$$\Gamma_{\rho, \theta} = \{re^{-i\theta} \in \mathbb{C} \mid r \geq \rho\} \cup \{\rho e^{i\phi} \in \mathbb{C} \mid \theta \leq \phi \leq 2\pi - \theta\} \cup \{re^{+i\theta} \in \mathbb{C} \mid r \geq \rho\},$$

where we denote $\Gamma_{0, \theta}$ simply by Γ_θ . Then, given any $A \in \mathcal{P}(\theta)$ we can define its complex powers A^z for $\operatorname{Re}(z) < 0$ by

$$A^z = \frac{1}{2\pi i} \int_{\Gamma_{\rho, \theta}} (-\lambda)^z (A + \lambda)^{-1} d\lambda \in \mathcal{L}(X_0),$$

for certain $\rho > 0$ due to Lemma 2.2. The above family together with $A^0 = I$ is a strongly continuous holomorphic semigroup on X_0 (see e.g. [1, Theorem III.4.6.2] and [1, Theorem III.4.6.5]). The definition can be extended to any $z \in \mathbb{C}$, and for $\operatorname{Re}(z) \geq 0$, A^z are in general unbounded operators (see e.g. [1, Theorem III.4.6.5]). Of particular interest from the PDEs' point of view is the case when the purely imaginary powers turn out to be bounded operators.

Definition 2.3 (Bounded imaginary powers). *Let $A \in \mathcal{P}(\theta)$, $\theta \in [0, \pi)$. We say that A has bounded imaginary powers if there exists some $\varepsilon > 0$ and $c_{A, \varepsilon} \geq 1$ such that*

$$A^{it} \in \mathcal{L}(X_0) \quad \text{and} \quad \|A^{it}\|_{\mathcal{L}(X_0)} \leq c_{A, \varepsilon} \quad \text{for all } t \in [-\varepsilon, \varepsilon].$$

In this case, (see e.g. [1, Corollary III.4.7.2]) $A^{it} \in \mathcal{L}(X_0)$ for all $t \in \mathbb{R}$ and there exists a $\phi \geq 0$, called power angle of A , such that $\|A^{it}\|_{\mathcal{L}(X_0)} \leq M_{A, \phi} e^{\phi|t|}$, $t \in \mathbb{R}$, with some $M_{A, \phi} \geq 1$, and we write $A \in \mathcal{BIP}(\phi)$.

Similarly to the complex powers, we can define the holomorphic functional calculus for a sectorial operator by replacing $(-\lambda)^z$ in the Dunford integral with an appropriate function. Then, the following basic property can be satisfied.

Definition 2.4 (Bounded H^∞ -functional calculus). *Let X_0 be a complex Banach space, $A \in \mathcal{P}(\theta)$, $\theta \in (0, \pi)$, and $\phi \in [0, \theta)$. Let $H_0^\infty(\phi)$ be the space of all bounded holomorphic functions $f : \mathbb{C} \setminus S_\phi \rightarrow \mathbb{C}$ such that*

$$|f(\lambda)| \leq c \left(\frac{|\lambda|}{1 + |\lambda|^2} \right)^\eta, \quad \text{for any } \lambda \in \mathbb{C} \setminus S_\phi,$$

with some $c > 0$ and $\eta > 0$ depending on f . Any $f \in H_0^\infty(\phi)$ defines an element $f(-A) \in \mathcal{L}(X_0)$ by

$$f(-A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} f(\lambda) (A + \lambda)^{-1} d\lambda.$$

We say that the operator A admits a bounded H^∞ -calculus of angle ϕ , and we denote by $A \in \mathcal{H}^\infty(\phi)$, if

$$\|f(-A)\|_{\mathcal{L}(X_0)} \leq C_{A, \phi} \sup_{\lambda \in \mathbb{C} \setminus S_\phi} |f(\lambda)|, \quad \text{for any } f \in H_0^\infty(\phi),$$

with some constant $C_{A, \phi} \geq 1$ that is called H^∞ -bound of A and depends only on A and ϕ .

By replacing the standard boundedness condition in the definition of sectoriality by the *Rademacher boundedness*, we obtain the following property for a sectorial operator.

Definition 2.5. *Let $\{\epsilon_k\}_{k=1}^\infty$ be the sequence of the Rademacher functions and $\theta \in [0, \pi)$. An operator $A \in \mathcal{P}(\theta)$ is called R -sectorial of angle θ , if for any choice of $\lambda_1, \dots, \lambda_N \in S_\theta \setminus \{0\}$, $x_1, \dots, x_N \in X_0$, $N \in \mathbb{N}$, we have that*

$$\left\| \sum_{k=1}^N \epsilon_k \lambda_k (A + \lambda_k)^{-1} x_k \right\|_{L^2(0,1; X_0)} \leq R_{A, \theta} \left\| \sum_{k=1}^N \epsilon_k x_k \right\|_{L^2(0,1; X_0)},$$

for some constant $R_{A, \theta} \geq 1$ that is called R -sectorial bound of A and depends on A and θ .

Let us now see how the above stated properties are applied to the theory of parabolic PDEs. By starting with the linear theory, consider the Cauchy problem

$$(2.3) \quad u'(t) + Au(t) = g(t), \quad t \in (0, T),$$

$$(2.4) \quad u(0) = 0$$

in the X_0 -valued L^q -space $L^q(0, T; X_0)$, where $q \in (1, \infty)$, $T > 0$ is finite and $-A : X_1 \rightarrow X_0$ is the infinitesimal generator of a bounded analytic semigroup on X_0 . The operator A has *maximal L^q -regularity* if for some q (and hence for all, according to a result of G. Dore) we have that for any $g \in L^q(0, T; X_0)$ there exists a unique $u \in W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ solving (2.3)-(2.4) that depends continuously on g . Note that, by setting $u(t) = e^{ct}v(t)$, $c > 0$, we can insert a positive shift c to the operator A in (2.3). Finally, recall the standard embedding of the maximal L^q -regularity space, namely

$$(2.5) \quad W^{1,q}(\tau, T; X_0) \cap L^q(\tau, T; X_1) \hookrightarrow C([\tau, T]; (X_1, X_0)_{\frac{1}{q}, q}), \quad \tau \in [0, T],$$

where the norm of the embedding is independent of τ and T (see e.g. [1, Theorem III.4.10.2] or [22, (2.6)]). Here $(X_0, X_1)_{\xi, q}$, $\xi \in (0, 1)$, denotes real interpolation.

All the spaces we consider in the sequel belong to the class of UMD (unconditionality of martingale differences property) spaces. By using the underline geometric properties of such spaces, we can relate the boundedness of the imaginary powers of an operator to the maximal L^q -regularity by the following classical result.

Theorem 2.6. (Dore and Venni, [7, Theorem 1]) *In a UMD Banach space, an operator $A \in \mathcal{BIP}(\phi)$ with $\phi < \frac{\pi}{2}$ has maximal L^q -regularity.*

However, in UMD spaces the R -sectoriality is weaker than the boundedness of the imaginary powers property (see [4, Theorem 4]) and it turns out to be sufficient for maximal L^q -regularity (it actually leads to the characterization of this property in UMD spaces due to [31, Theorem 4.2]).

Theorem 2.7. (Kalton and Weis, [12, Theorem 6.5]) *In a UMD Banach space any R -sectorial operator of angle θ with $\theta > \frac{\pi}{2}$ has maximal L^q -regularity.*

We state next an elementary boundedness property of holomorphic semigroups, which we use for later estimates. Most of the following result can be found in [1, Section IV.2.1].

Proposition 2.8. *Let X_0 be a complex Banach space, $A \in \mathcal{P}(\theta)$ with $\theta > \frac{\pi}{2}$, and let $K_{A, \theta} > 1$ be a sectorial bound for A . Then, A generates a bounded holomorphic semigroup $\{e^{-wA}\}_{w \in S_\phi \setminus \{0\}}$ on X_0 , where $\phi \in [0, \theta - \frac{\pi}{2})$. Furthermore, for any $a \in [0, 1)$ and $t > 0$ we have that $e^{-tA} \in C((0, \infty); \mathcal{L}(X_0, \mathcal{D}(A^a)))$ and*

$$\|A^a e^{-tA}\|_{\mathcal{L}(X_0)} \leq Ct^{-a}, \quad t > 0,$$

for some constant C depending only on $K_{A, \theta}$, θ and a .

Proof. If $A \in \mathcal{P}(\theta)$ with $\theta > \frac{\pi}{2}$, then by the characterization result [2, Theorem 3.7.11] and the formula [2, (3.46)], A generates a bounded holomorphic semigroup on X_0 given by the following integral

$$(2.6) \quad e^{-wA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda w} (A + \lambda)^{-1} d\lambda, \quad w \in S_{\theta - \frac{\pi}{2}} \setminus \partial S_{\theta - \frac{\pi}{2}}.$$

Assume first that $a \in (0, 1)$. Recall that by $\mathcal{D}(A^a)$ we denote the Banach space $(\mathcal{D}(A^a), \|A^a \cdot\|_{X_0})$. Since the integral

$$\int_{\Gamma_\theta} e^{\lambda t} A^a (A + \lambda)^{-1} d\lambda$$

converges absolutely, we have that $e^{-tA} \in \mathcal{L}(X_0, \mathcal{D}(A^a))$ and

$$A^a e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} A^a (A + \lambda)^{-1} d\lambda, \quad t > 0.$$

Therefore, for sufficiently small $c > 0$ and $\varepsilon \in (0, \theta - \frac{\pi}{2})$, by Lemma 2.2, Cauchy's theorem and Fubini's theorem, we obtain that

$$\begin{aligned}
A^a e^{-tA} &= \frac{1}{2\pi i} \int_{c+\Gamma_{\theta-\varepsilon}} e^{\lambda t} A(A+\lambda)^{-1} A^{a-1} d\lambda \\
&= \frac{1}{2\pi i} \int_{c+\Gamma_{\theta-\varepsilon}} e^{\lambda t} A(A+\lambda)^{-1} \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} (-z)^{a-1} (A+z)^{-1} dz \right) d\lambda \\
&= \left(\frac{1}{2\pi i} \right)^2 \int_{c+\Gamma_{\theta-\varepsilon}} \int_{\Gamma_\theta} \frac{(-z)^{a-1} e^{\lambda t}}{z-\lambda} A((A+\lambda)^{-1} - (A+z)^{-1}) dz d\lambda \\
&= \left(\frac{1}{2\pi i} \right)^2 \int_{c+\Gamma_{\theta-\varepsilon}} \int_{\Gamma_\theta} \frac{(-z)^{a-1} e^{\lambda t}}{z-\lambda} A(A+\lambda)^{-1} dz d\lambda \\
&\quad + \left(\frac{1}{2\pi i} \right)^2 \int_{\Gamma_\theta} \int_{c+\Gamma_{\theta-\varepsilon}} \frac{(-z)^{a-1} e^{\lambda t}}{\lambda-z} A(A+z)^{-1} d\lambda dz.
\end{aligned}$$

By Cauchy's theorem, the first term on the right hand side of the above equation is equal to zero. Hence, we conclude that

$$A^a e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-z)^{a-1} e^{zt} A(A+z)^{-1} dz.$$

By changing variables in the above integral, we get that

$$A^a e^{-tA} = t^{-a} \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^{a-1} e^{\lambda} A(A + \frac{\lambda}{t})^{-1} d\lambda.$$

The result now follows by the above expression and the dominated convergence theorem.

Consider now the case of $a = 0$. By changing variables to (2.6) we obtain that

$$e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} \frac{e^z}{t} (A + \frac{z}{t})^{-1} dz, \quad t > 0.$$

Therefore, by Cauchy's theorem for any $\rho > 0$ we obtain that

$$e^{-tA} = \frac{1}{2\pi i} \int_{-\Gamma_{\rho, \pi-\theta}} \frac{e^z}{z} \frac{z}{t} (A + \frac{z}{t})^{-1} dz, \quad t > 0,$$

where $-\Gamma_{\rho, \pi-\theta} = \{e^{i\pi}\lambda \mid \lambda \in \Gamma_{\rho, \pi-\theta}\}$, and the result follows by the sectoriality of A . \square

Next we state the well known version of Theorem 2.7 for non-zero initial data. We emphasize at the norm estimates of the solution operator.

Theorem 2.9. *Let $X_1 \xrightarrow{d} X_0$ be a continuously and densely injected Banach couple, $\theta \in (\frac{\pi}{2}, \pi)$ and $A : \mathcal{D}(A) = X_1 \mapsto X_0$ be an R -sectorial operator of angle θ with R -sectorial bound $R_{A, \theta}$ and sectorial bound $K_{A, \theta}$. Assume that X_0 is UMD and take $g \in L^q(0, \infty; X_0)$ and $u_0 \in (X_1, X_0)_{\frac{1}{q}, q}$, for some $q \in (1, \infty)$. Then, for any $T > 0$ the Cauchy problem*

$$(2.7) \quad u'(t) + Au(t) = g(t), \quad t \in (0, T),$$

$$(2.8) \quad u(0) = u_0$$

has a unique solution $u \in W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ and the following estimate holds

$$(2.9) \quad \|u\|_{W^{1,q}(0, T; X_0)} + \|u\|_{L^q(0, T; X_1)} \leq C(\|g\|_{L^q(0, T; X_0)} + \|u_0\|_{(X_1, X_0)_{\frac{1}{q}, q}}),$$

with some constant $C > 0$ depending only on the data $\{A, K_{A, \theta}, R_{A, \theta}, \theta, q\}$.

Furthermore, if $g \in L^\infty(0, \infty; (X_0, X_1)_{\phi, \eta})$ for some $\phi \in (0, 1)$ and $\eta \in (1, \infty)$, then $u(t) \in X_1$ for each $t > 0$, and for any $\alpha \in (0, \phi)$ and $\beta \in (-1, -\frac{1}{q})$ the following estimate holds

$$(2.10) \quad \|u(t)\|_{X_1} \leq Mt^\alpha \|g\|_{L^\infty(0, \infty; (X_0, X_1)_{\phi, \eta})} + Lt^\beta \|u_0\|_{(X_1, X_0)_{\frac{1}{q}, q}}, \quad t > 0,$$

with some constants $M > 0$ and $L > 0$ depending only on the data $\{A, K_{A, \theta}, \theta, \phi, \eta, \alpha\}$ and $\{A, K_{A, \theta}, \theta, q, \beta\}$ respectively.

Proof. Consider the following two linear parabolic problems

$$(2.11) \quad u_1'(t) + Au_1(t) = g(t), \quad t \in (0, T),$$

$$(2.12) \quad u_1(0) = 0$$

and

$$(2.13) \quad u_2'(t) + Au_2(t) = 0, \quad t \in (0, T),$$

$$(2.14) \quad u_2(0) = u_0.$$

For any $T > 0$, by Theorem 2.7, the problem (2.11)-(2.12) admits a unique solution $u_1 \in W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$. More precisely, let us denote again by A the natural extension of A from X_0 to $L^q(0, T; X_0)$, i.e. $(Af)(t) = Af(t)$ for almost all t . Clearly, A with domain $L^q(0, T; X_1)$ in $L^q(0, T; X_0)$ is also R -sectorial of angle θ . Further, denote by B the operator $u \mapsto \partial_t u$ in $L^q(0, T; X_0)$ with domain

$$\mathcal{D}(B) = \{u \in W^{1,q}(0, T; X_0) \mid u(0) = 0\}.$$

Recall that the resolvent $(B + \lambda)^{-1}$ of the above operator is defined for any $\lambda \in \mathbb{C}$ and acts by

$$(2.15) \quad (B + \lambda)^{-1}g(t) = \int_0^t e^{(s-t)\lambda} g(s) ds, \quad g \in L^q(0, T; X_0).$$

Since our underline Banach space fulfils the UMD property, by [9, Theorem 8.5.8], for any $\phi \in (0, \frac{\pi}{2})$ the operator B has bounded H^∞ -calculus of angle ϕ . Further, if in the definition formula for the H^∞ -calculus for B we replace $(B + \lambda)^{-1}g(t)$ by

$$\int_0^t e^{(s-t)\lambda} g(s) \psi_{[0, T]}(s) ds,$$

where $\psi_{[0, T]}$ denotes the characteristic function on $[0, T]$, then by [9, Corollary 8.5.3 (a)] we find that for each T the H^∞ -bound of B can be chosen to be bounded by the H^∞ -bound for the case of $T = \infty$. Therefore, we conclude that the H^∞ -bound of B can be chosen to be uniformly bounded in $T \in (0, \infty)$.

The operators A and B are resolvent commuting in the sense of [1, (III.4.9.1)]. Hence, by [12, Theorem 6.3] the sum $A + B$ with domain $W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ in $L^q(0, T; X_0)$ is closed and invertible. By [6, Theorem 3.7] (or alternatively by [24, Theorem 2.1]) its inverse is given by

$$(2.16) \quad (A + B)^{-1} = \frac{1}{2\pi i} \int_{\Gamma_\theta} (A + \lambda)^{-1} (B - \lambda)^{-1} d\lambda \in \mathcal{L}(X_0).$$

Further, by the estimates in the proof of [12, Theorem 4.4] (alternatively see [10, Theorem 2.1] or the proof of [23, Theorem 3.1]), the $\mathcal{L}(L^q(0, T; X_0))$ -norm of $B(A + B)^{-1}$ can be estimated by the H^∞ -bound of B and the R -sectorial bound of A , and hence it is uniformly bounded in T . Therefore, by recalling that $X_1 = (X_1, \|A \cdot\|_{X_0})$, we estimate

$$\begin{aligned} & \|u_1\|_{W^{1,q}(0, T; X_0)} + \|u_1\|_{L^q(0, T; X_1)} \\ &= \|u_1\|_{L^q(0, T; X_0)} + \|Bu_1\|_{L^q(0, T; X_0)} + \|Au_1\|_{L^q(0, T; X_0)} \\ &= \|(A + B)^{-1}g\|_{L^q(0, T; X_0)} + \|B(A + B)^{-1}g\|_{L^q(0, T; X_0)} + \|A(A + B)^{-1}g\|_{L^q(0, T; X_0)} \\ (2.17) \quad & \leq C_1 \|g\|_{L^q(0, T; X_0)}, \end{aligned}$$

for some constant $C_1 > 0$ depending only on the data $\{A, K_{A, \theta}, R_{A, \theta}, \theta, q\}$.

Concerning the problem (2.13), since $A \in \mathcal{P}(\theta)$, $\theta > \frac{\pi}{2}$, by the characterization result [2, Theorem 3.7.11] and the formula [2, (3.46)], $-A$ generates a bounded holomorphic semigroup on X_0 which on the positive real semiaxis is given by the following formula

$$(2.18) \quad e^{-tA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} (A + \lambda)^{-1} d\lambda, \quad t > 0.$$

Since the integral

$$\int_{\Gamma_\theta} e^{\lambda t} A(A + \lambda)^{-1} d\lambda$$

converges absolutely when $t > 0$, by (2.18) we have that e^{-tA} maps X_0 to $\mathcal{D}(A)$ and by the dominated convergence theorem

$$(2.19) \quad Ae^{-tA} = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{\lambda t} A(A + \lambda)^{-1} d\lambda \in C((0, \infty); \mathcal{L}(X_0)).$$

Therefore, if $w \in (X_1, X_0)_{\frac{1}{q}, q}$ and $\sigma = \frac{q-1}{2q}$, by [1, (I.2.5.2)], [1, (I.2.9.6)] and [1, Theorem III.4.6.5] we have that $Ae^{-tA}w = A^{1-\sigma}e^{-tA}A^\sigma w$. Then, for any $\tau > 1$ by Proposition 2.8 we estimate

$$(2.20) \quad \begin{aligned} & \|Ae^{-tA}w\|_{L^q(1, \tau; X_0)} \\ & \leq \|A^{1-\sigma}e^{-tA}\|_{L^q(1, \tau; \mathcal{L}(X_0))} \|A^\sigma w\|_{X_0} \\ & \leq C_2(1 - \tau^{\frac{1-q}{2}})^{\frac{1}{q}} \|w\|_{(X_1, X_0)_{\frac{1}{q}, q}}, \end{aligned}$$

with some constant $C_2 > 0$ depending only on the data $\{A, K_{A, \theta}, \theta, q\}$.

For any $q \in (1, \infty)$, let the space

$$D_A(1 - \frac{1}{q}, q) = \left\{ w \in X_0 \mid t \mapsto Ae^{-tA}w \in L^q(0, 1; X_0) \right\}$$

with norm $\|w\|_{D_A(1-\frac{1}{q}, q)} = \|w\|_{X_0} + \|Ae^{-tA}w\|_{L^q(0, 1; X_0)}$. By the characterization result [17, Proposition 2.2.2], we have that $D_A(1 - \frac{1}{q}, q) = (X_1, X_0)_{\frac{1}{q}, q}$ up to norm equivalence. Hence, there exists some constant $C_3 > 0$ depending only on A and q such that

$$(2.21) \quad \|Ae^{-tA}w\|_{L^q(0, 1; X_0)} \leq C_3 \|w\|_{(X_1, X_0)_{\frac{1}{q}, q}}$$

for any $w \in (X_1, X_0)_{\frac{1}{q}, q}$.

Therefore, by (2.20) and (2.21), $u_2 = e^{-tA}u_0$ is a $W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ -solution of (2.13)-(2.14). Moreover, we have that

$$(2.22) \quad \begin{aligned} & \|u_2\|_{W^{1,q}(0, T; X_0)} + \|u_2\|_{L^q(0, T; X_1)} \\ & = \|u_2\|_{L^q(0, T; X_0)} + 2\|Au_2\|_{L^q(0, T; X_0)} \\ & \leq C_4 \|u_0\|_{(X_1, X_0)_{\frac{1}{q}, q}}, \end{aligned}$$

for some constant $C_4 > 0$ depending only on the data $\{A, K_{A, \theta}, \theta, q\}$.

Returning to the original problem, we note that $u = u_1 + u_2$ is a $W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ -solution of (2.7)-(2.8) and the required estimate follows by (2.17) and (2.22). Finally, the uniqueness of u follows by the injectivity of $A + B$ and the fact that the difference f of two possible solutions belongs to the domain of $A + B$ and satisfies $(A + B)f = 0$.

Concerning the estimate (2.10), if $\alpha \in (0, 1)$, by (2.16), Lemma 2.2, Cauchy's theorem and Fubini's theorem we find that

$$\begin{aligned} & (A + B)^{-1}A^{-\alpha} \\ & = \left(\frac{1}{2\pi i}\right)^2 \int_{-c+\Gamma_\theta} (A + \lambda)^{-1}(B - \lambda)^{-1} \left(\int_{\Gamma_{\theta-\delta}} (-z)^{-\alpha} (A + z)^{-1} dz \right) d\lambda \\ & = \left(\frac{1}{2\pi i}\right)^2 \int_{-c+\Gamma_\theta} \int_{\Gamma_{\theta-\delta}} \frac{(-z)^{-\alpha}}{z - \lambda} (B - \lambda)^{-1} ((A + \lambda)^{-1} - (A + z)^{-1}) dz d\lambda \\ & = \left(\frac{1}{2\pi i}\right)^2 \int_{-c+\Gamma_\theta} \int_{\Gamma_{\theta-\delta}} \frac{(-z)^{-\alpha}}{z - \lambda} (B - \lambda)^{-1} (A + \lambda)^{-1} dz d\lambda \\ & \quad + \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_{\theta-\delta}} \int_{-c+\Gamma_\theta} \frac{(-z)^{-\alpha}}{\lambda - z} (B - \lambda)^{-1} (A + z)^{-1} d\lambda dz, \end{aligned}$$

for certain $c, \delta > 0$ sufficiently small. The second integral on the right hand side of the above equation equals to zero. Therefore,

$$(2.23) \quad (A + B)^{-1}A^{-\alpha} = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^{-\alpha} (A + \lambda)^{-1} (B - \lambda)^{-1} d\lambda.$$

If $g \in L^\infty(0, \infty; (X_0, X_1)_{\phi, \eta})$ for some $\phi \in (0, 1)$ and $\eta \in (1, \infty)$, then by restricting $a \in (0, \phi)$, by [1, (I.2.5.2)] and [1, (I.2.9.6)] we have that

$$g \in L^\infty(0, \infty; (X_0, X_1)_{\phi, \eta}) \hookrightarrow L^\infty(0, \infty; \mathcal{D}(A^\alpha)).$$

Therefore, (2.23) implies

$$u_1 = (A + B)^{-1} A^{-\alpha} A^\alpha g = \frac{1}{2\pi i} \int_{\Gamma_\theta} (-\lambda)^{-\alpha} (A + \lambda)^{-1} (B - \lambda)^{-1} A^\alpha g d\lambda.$$

Then, by (2.15) and Fubini's theorem for each $t > 0$ we have that

$$u_1 = \int_0^t \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^{(t-s)\lambda} (-\lambda)^{-\alpha} (A + \lambda)^{-1} A^\alpha g(s) d\lambda \right) ds \in X_0.$$

Thus, by changing variables we get that

$$(2.24) \quad u_1 = \int_0^t \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^z (-z)^{-\alpha} \left(A + \frac{z}{t-s} \right)^{-1} dz \right) (t-s)^{\alpha-1} A^\alpha g(s) ds.$$

Since the integral

$$\int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{\rho} \right)^{-1} dz$$

converges absolutely uniformly in $\rho > 0$, by the resolvent formula and the dominated convergence theorem, it easily follows that

$$\rho \mapsto \int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{\rho} \right)^{-1} dz \in C((0, \infty); \mathcal{L}(X_0)) \cap L^\infty(0, \infty; \mathcal{L}(X_0)).$$

Therefore, (2.24) implies

$$(2.25) \quad u_1(t) = \int_0^t A^{-1} \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{t-s} \right)^{-1} dz \right) (t-s)^{\alpha-1} A^\alpha g(s) ds \in X_0,$$

for all $t > 0$. Now, since the integral

$$\int_0^t \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{t-s} \right)^{-1} dz \right) (t-s)^{\alpha-1} A^\alpha g(s) ds$$

converges absolutely, by (2.25) we find that $u_1(t) \in X_1$ for each $t > 0$ and

$$Au_1(t) = \int_0^t \left(\frac{1}{2\pi i} \int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{t-s} \right)^{-1} dz \right) (t-s)^{\alpha-1} A^\alpha g(s) ds \in X_0, \quad t > 0.$$

Furthermore, for any $t > 0$ we estimate

$$\begin{aligned} & \|Au_1(t)\|_{X_0} \\ & \leq \left(\int_0^t \frac{1}{2\pi} \left\| \int_{\Gamma_\theta} e^z (-z)^{-\alpha} A \left(A + \frac{z}{t-s} \right)^{-1} dz \right\|_{\mathcal{L}(X_0)} (t-s)^{\alpha-1} ds \right) \|A^\alpha g\|_{L^\infty(0, t; X_0)} \\ (2.26) \quad & \leq C_5 t^\alpha \|g\|_{L^\infty(0, \infty; (X_0, X_1)_{\phi, \eta})}, \end{aligned}$$

for some constant $C_5 > 0$ depending only on the data $\{A, K_{A, \theta}, \theta, \phi, \eta, \alpha\}$.

Concerning the part u_2 , by (2.19) we have that $u_2 \in C((0, \infty); X_1)$. By taking $\gamma \in (0, 1 - \frac{1}{q})$, due to [1, (I.2.5.2)], [1, (I.2.9.6)] and [1, Theorem III.4.6.5] for any $t > 0$ we write

$$Au_2(t) = A^{1-\gamma} e^{-tA} A^\gamma u_0.$$

Therefore, by Proposition 2.8 we estimate

$$\|Au_2(t)\|_{X_0} \leq C_6 t^{\gamma-1} \|A^\gamma u_0\|_{X_0} \leq C_7 t^{\gamma-1} \|u_0\|_{(X_1, X_0)_{\frac{1}{q}, q}}, \quad t > 0,$$

for some constants $C_6 > 0$ and $C_7 > 0$ depending only on the data $\{K_{A, \theta}, \theta, \gamma\}$ and $\{A, K_{A, \theta}, \theta, q, \gamma\}$ respectively. Hence, (2.10) follows by the above inequality and (2.26). \square

Corollary 2.10. *Let $X_1 \xrightarrow{d} X_0$ be a continuously and densely injected Banach couple, $\theta \in (\frac{\pi}{2}, \pi)$, $c > 0$ and $A : \mathcal{D}(A) = X_1 \mapsto X_0$ be closed linear operator such that $A + c$ is R -sectorial of angle θ with R -sectorial bound $R_{A,\theta,c}$. Assume that X_0 is UMD and take $g \in L^q(0, \infty; X_0)$ and $u_0 \in (X_1, X_0)_{\frac{1}{q}, q}$, for some $q \in (1, \infty)$. Then, for any $T > 0$ the Cauchy problem*

$$(2.27) \quad u'(t) + Au(t) = g(t), \quad t \in (0, T),$$

$$(2.28) \quad u(0) = u_0$$

has a unique solution $u \in W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ and the following estimate holds

$$(2.29) \quad \|u\|_{W^{1,q}(0,T;X_0)} + \|u\|_{L^q(0,T;X_1)} \leq (c+1)e^{cT}C(\|g\|_{L^q(0,T;X_0)} + \|u_0\|_{(X_1,X_0)_{\frac{1}{q},q}}),$$

with some constant $C > 0$ depending only on the data $\{A, R_{A,\theta,c}, \theta, q, c\}$.

Proof. Follows by setting $u(t) = e^{ct}f(t)$ to (2.27)-(2.28) and then applying Theorem 2.9 and (2.9) to the resulting problem, with the sectorial bound chosen to be equal to

$$2 \max \left\{ R_{A,\theta,c}, \sup_{\lambda \in S_\theta, |\lambda| \leq 1} \|(A + c + \lambda)^{-1}\|_{\mathcal{L}(X_0)} \right\}.$$

□

We choose to treat our semilinear problem by regarding it as a quasilinear parabolic equation. Hence, we consider problems of the form

$$(2.30) \quad u'(t) + A(u(t))u(t) = G(t, u(t)) + g(t), \quad t \in (0, T_0),$$

$$(2.31) \quad u(0) = u_0$$

in $L^q(0, T_0; X_0)$ with $q \in (1, \infty)$ and $T_0 > 0$ finite. Maximal L^q -regularity for the solution of the linearized problem together with appropriate Lipschitz continuity conditions imply existence, uniqueness and maximal L^q -regularity for the short time solution of the original problem, as the following well known theorem guarantees.

Theorem 2.11. (Clément and Li, [3, Theorem 2.1]) *Let $u_0 \in (X_1, X_0)_{\frac{1}{q}, q}$ and assume that there exists an open neighborhood $U \subset (X_1, X_0)_{\frac{1}{q}, q}$ of u_0 such that:*

- (H1) $A(u_0) : X_1 \rightarrow X_0$ has maximal L^q -regularity,
- (H2) $A \in C^{1-}(U; \mathcal{L}(X_1, X_0))$,
- (H3) $G \in C^{1-, 1-}([0, T_0] \times U; X_0)$,
- (H4) $g \in L^q(0, T_0; X_0)$.

Then, there exists a $T \in (0, T_0)$ and a unique $u \in W^{1,q}(0, T; X_0) \cap L^q(0, T; X_1)$ solving the equation (2.30)-(2.31) on $(0, T)$.

The above theorem will be the main tool for obtaining short time solution and deriving regularity results in the sequel.

3. GENERAL THEORY FOR CONE DIFFERENTIAL OPERATORS

Let \mathcal{B} be an $(n+1)$ -dimensional, $n \geq 1$, smooth connected manifold with possibly disconnected closed (i.e. compact without boundary) boundary $\partial\mathcal{B}$. We endow \mathcal{B} with a Riemannian metric \mathfrak{g} such that when it is restricted to a collar neighborhood $[0, 1) \times \partial\mathcal{B}$ of the boundary it admits the warped product structure

$$(3.32) \quad \mathfrak{g}|_{[0,1) \times \partial\mathcal{B}} = dx^2 + x^2 \mathfrak{h}(x),$$

where $x \in [0, 1)$ and the map $x \mapsto \mathfrak{h}(x)$ is a smooth up to $x = 0$ family of Riemannian metrics on the cross section $\partial\mathcal{B}$ that does not degenerate up to $x = 0$. We call $\mathbb{B} = (\mathcal{B}, \mathfrak{g})$ *conic manifold* or *manifold with (warped) conical singularities*, which are identified with the subset $\{0\} \times \partial\mathcal{B}$ of \mathcal{B} . For any $x \in [0, 1)$, denote $\partial\mathbb{B} = (\partial\mathcal{B}, \mathfrak{h}(x))$. If \mathfrak{h} is independent of x , then we have straight conical tips.

The naturally appearing differential operators on a conic manifold are *conically degenerate* differential operators and belong to the class of *cone differential operators* or *Fuchs type operators*. An μ -th order cone differential operator A , $\mu \in \mathbb{N}$, is an μ -th order differential operator with smooth

coefficients in the interior \mathcal{B}° of \mathcal{B} such that near the boundary, e.g. in the collar neighborhood $(0, 1) \times \partial\mathcal{B}$, it admits the following form in terms of *Mellin derivatives*, namely

$$A = x^{-\mu} \sum_{k=0}^{\mu} a_k(x)(-x\partial_x)^k, \quad \text{where } a_k \in C^\infty([0, 1]; \text{Diff}^{\mu-k}(\partial\mathcal{B})).$$

By freezing the coefficients of A at the boundary we obtain the model cone operator \hat{A} of A given by

$$\hat{A} = x^{-\mu} \sum_{k=0}^{\mu} a_k(0)(-x\partial_x)^k,$$

which acts on the infinite half cylinder $\mathbb{R}_+ \times \partial\mathcal{B}$.

Beyond the standard principal pseudodifferential symbol $\sigma_\psi^\mu(A)$, we associate two further symbols to a cone differential operator. Hence, if $(x, y) \in [0, 1] \times \partial\mathcal{B}$ are local coordinates near the boundary and (ξ, η) are the corresponding covariables, the *rescaled symbol* is defined by

$$\tilde{\sigma}_\psi^\mu(A)(y, \eta, \xi) = \sum_{k=0}^{\mu} \sigma_\psi^{\mu-k}(a_k)(0, y, \eta)(-i\xi)^k \in C^\infty((T^*\partial\mathcal{B} \times \mathbb{R}) \setminus \{0\}).$$

Further, the *conormal symbol* is given by the following holomorphic family of differential operators defined on the boundary, namely

$$\sigma_M^\mu(A)(z) = \sum_{k=0}^{\mu} a_k(0)z^k : \mathbb{C} \mapsto \mathcal{L}(H_p^s(\partial\mathbb{B}), H_p^{s-\mu}(\partial\mathbb{B})),$$

where $s \in \mathbb{R}$, $p \in (1, \infty)$ and $H_p^s(\partial\mathbb{B})$ denotes the usual Sobolev space. A cone differential operator A is called \mathbb{B} -elliptic if $\sigma_\psi^\mu(A)$ is invertible on $T^*\mathcal{B}^\circ$ and $\tilde{\sigma}_\psi^\mu(A)$ is pointwise invertible up to $x = 0$. Later on, this will be the case for the Laplacian Δ .

Cone differential operators act naturally on scales of weighted *Mellin-Sobolev spaces* $\mathcal{H}_p^{s, \gamma}(\mathbb{B})$, $s, \gamma \in \mathbb{R}$, $p \in (1, \infty)$, i.e. any such operator A of order μ induces a bounded map

$$A : \mathcal{H}_p^{s+\mu, \gamma+\mu}(\mathbb{B}) \rightarrow \mathcal{H}_p^{s, \gamma}(\mathbb{B}).$$

If $s \in \mathbb{N}_0$, then $\mathcal{H}_p^{s, \gamma}(\mathbb{B})$ is the space of all functions u in $H_{p, \text{loc}}^s(\mathbb{B}^\circ)$ such that, near the boundary

$$x^{\frac{n+1}{2}-\gamma}(x\partial_x)^k \partial_y^\alpha (\omega(x, y)u(x, y)) \in L^p([0, 1] \times \partial\mathcal{B}, \sqrt{\det(\mathfrak{h}(x))} \frac{dx}{x} dy), \quad k + |\alpha| \leq s,$$

where ω is a cut-off function near $\{0\} \times \partial\mathcal{B}$, i.e. a smooth non-negative function ω on \mathcal{B} with $\omega = 1$ near $\{0\} \times \partial\mathcal{B}$ and $\omega = 0$ outside the collar neighborhood $[0, 1] \times \partial\mathcal{B}$ of the boundary. In general we have the following.

Definition 3.1. For any $\gamma \in \mathbb{R}$ consider the map

$$M_\gamma : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \rightarrow C_0^\infty(\mathbb{R}^{n+1}) \quad \text{defined by } u(x, y) \mapsto e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, y).$$

Further, take a covering $\kappa_i : U_i \subseteq \partial\mathcal{B} \rightarrow \mathbb{R}^n$, $i \in \{1, \dots, N\}$, of $\partial\mathcal{B}$ by coordinate charts and let $\{\phi_i\}_{i \in \{1, \dots, N\}}$ be a subordinated partition of unity. For any $s \in \mathbb{R}$ and $p \in (1, \infty)$ let $\mathcal{H}_p^{s, \gamma}(\mathbb{B})$ be the space of all distributions u in \mathbb{B}° such that

$$\|u\|_{\mathcal{H}_p^{s, \gamma}(\mathbb{B})} = \sum_{i=1}^N \|M_\gamma(1 \otimes \kappa_i)_*(\omega \phi_i u)\|_{H_p^s(\mathbb{R}^{n+1})} + \|(1 - \omega)u\|_{H_p^s(\mathbb{B})}$$

is defined and finite, where ω is a fixed cut-off function near $\{0\} \times \partial\mathcal{B}$ and $*$ refers to the push-forward of distributions.

The above space is independent of the choice of ω and $\{\kappa_i\}_{i \in \{1, \dots, N\}}$. Furthermore, since the usual Sobolev spaces are UMD, by [1, Theorem III.4.5.2], the Mellin-Sobolev spaces are also UMD.

We recall next some basic properties of Mellin-Sobolev spaces, that we use later.

Lemma 3.2. For any $p \in (1, \infty)$, $\gamma \in \mathbb{R}$ and $s > \frac{n+1}{p}$ there exists some $C > 0$ such that

$$\|uv\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} \leq C \|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} \|v\|_{\mathcal{H}_p^{s,\frac{n+1}{2}}(\mathbb{B})}.$$

In particular, if $s > \frac{n+1}{p}$ and $\gamma \geq \frac{n+1}{2}$, then $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is a Banach algebra under the choice of an equivalent norm. Furthermore, if $s > \frac{n+1}{p}$ then any function $u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is continuous in \mathcal{B}° and satisfies

$$|u(x, y)| \leq Lx^{\gamma - \frac{n+1}{2}} \|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})}$$

in $(0, 1) \times \partial\mathcal{B}$ for some constant $L > 0$.

Proof. This is [25, Corollary 2.8] and [25, Corollary 2.9]. \square

The following multiplication property is of particular interest.

Lemma 3.3. Let $p, q \in (1, \infty)$, $\gamma \in \mathbb{R}$ and $\sigma > 0$. Then, for any $s \in (-\sigma, \sigma)$ multiplication by an element in $\mathcal{H}_q^{\sigma + \frac{n+1}{q}, \frac{n+1}{2}}(\mathbb{B})$ defines a bounded map in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$. More precisely, if $u \in \mathcal{H}_q^{\sigma + \frac{n+1}{q}, \frac{n+1}{2}}(\mathbb{B})$ and $v \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ with $s \in (-\sigma, \sigma)$, then there exists some constant $C > 0$ such that

$$\|uv\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} \leq C \|u\|_{\mathcal{H}_q^{\sigma + \frac{n+1}{q}, \frac{n+1}{2}}(\mathbb{B})} \|v\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})}.$$

Proof. This is [26, Corollary 3.3]. \square

Finally we recall the following interpolation properties of Mellin-Sobolev spaces.

Lemma 3.4. Let $p, q \in (1, \infty)$, $s_0, s_1, \gamma_0, \gamma_1 \in \mathbb{R}$ and $\theta \in (0, 1)$. Then, for any $\varepsilon > 0$ we have the following embedding

$$\mathcal{H}_p^{s+\varepsilon, \gamma+\varepsilon}(\mathbb{B}) \hookrightarrow (\mathcal{H}_p^{s_0, \gamma_0}(\mathbb{B}), \mathcal{H}_p^{s_1, \gamma_1}(\mathbb{B}))_{\theta, q} \hookrightarrow \mathcal{H}_p^{s-\varepsilon, \gamma-\varepsilon}(\mathbb{B}),$$

where $s = (1 - \theta)s_0 + \theta s_1$ and $\gamma = (1 - \theta)\gamma_0 + \theta\gamma_1$. Furthermore, for the complex interpolation we have that

$$[\mathcal{H}_p^{s_0, \gamma_0}(\mathbb{B}), \mathcal{H}_p^{s_1, \gamma_1}(\mathbb{B})]_\theta = \mathcal{H}_p^{s, \gamma}(\mathbb{B}).$$

Proof. This is [26, Lemma 3.5], [26, Lemma 3.6] and [26, Lemma 3.7]. \square

We consider a cone differential operator A as an unbounded operator in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ with domain $C_0^\infty(\mathbb{B}^\circ)$, where C_0^∞ denotes smooth compactly supported functions. If A is \mathbb{B} -elliptic, then the domain of its minimal extension (i.e. its closure) $\underline{A}_{s,\min}$ is given by

$$\mathcal{D}(\underline{A}_{s,\min}) = \left\{ u \in \bigcap_{\varepsilon > 0} \mathcal{H}_p^{s+\mu, \gamma+\mu-\varepsilon}(\mathbb{B}) \mid x^{-\mu} \sum_{k=0}^{\mu} a_k(0)(-x\partial_x)^k(\omega u) \in \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \right\}.$$

If in addition the conormal symbol of A is invertible on $\{z \in \mathbb{C} \mid \operatorname{Re}(z) = \frac{n+1}{2} - \gamma - \mu\}$, then

$$\mathcal{D}(\underline{A}_{s,\min}) = \mathcal{H}_p^{s+\mu, \gamma+\mu}(\mathbb{B}).$$

For the domain of the maximal extension $\underline{A}_{s,\max}$, which is defined by

$$\mathcal{D}(\underline{A}_{s,\max}) = \left\{ u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \mid Au \in \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \right\},$$

we have that

$$(3.33) \quad \mathcal{D}(\underline{A}_{s,\max}) = \mathcal{D}(\underline{A}_{s,\min}) \oplus \mathcal{E}_{A,\gamma},$$

where $\mathcal{E}_{A,\gamma}$ is a finite-dimensional space independent of s , that is called *asymptotics space*, and consists of linear combinations of smooth functions on \mathcal{B} such that on $(0, 1) \times \partial\mathcal{B}$ they are of the form $c(y)x^{-\rho} \log^k(x)$, with $c \in C^\infty(\partial\mathcal{B})$, $\rho \in \mathbb{C}$ and $k \in \mathbb{N}$ (see [8], [16] or [27] for further details). Here, the powers ρ are determined explicitly by the poles in the strip $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \in [\frac{n+1}{2} - \gamma - \mu, \frac{n+1}{2} - \gamma]\}$ of the recursively defined family of symbols

$$(3.34) \quad g_0 = f_0^{-1}, \quad g_k = -(T^{-k}f_0^{-1}) \sum_{i=0}^{k-1} (T^{-i}f_{k-i})g_i, \quad k \in \{1, \dots, \mu-1\},$$

where

$$f_\nu(\lambda) = \frac{1}{\nu!} \sum_{i=0}^{\mu} (\partial_x^\nu a_i)(0) \lambda^i, \quad \nu \in \{0, \dots, \mu-1\},$$

and by T^σ , $\sigma \in \mathbb{R}$, we denote the action $(T^\sigma f)(\lambda) = f(\lambda + \sigma)$ (see e.g. [27, (2.7)-(2.8)]). The appearance of the logarithmic powers $\log^k(x)$ is related to the multiplicity of each pole. Therefore, the domain of each possible closed extension of A corresponds to a subspace of $\mathcal{E}_{A,\gamma}$.

4. THE LAPLACIAN ON A CONIC MANIFOLD

The Laplacian Δ induced by the metric (3.32), in the collar part $(0, 1) \times \partial\mathcal{B}$ has the following form

$$(4.35) \quad \Delta = x^{-2} ((x\partial_x)^2 + (n-1+H(x))(x\partial_x) + \Delta_{\mathfrak{h}(x)}),$$

where $\Delta_{\mathfrak{h}(x)}$ is the Laplacian on $\partial\mathcal{B}$ with respect to the metric $\mathfrak{h}(x)$ and

$$H(x) = \frac{x\partial_x(\det(\mathfrak{h}(x)))}{2\det(\mathfrak{h}(x))}.$$

Clearly, Δ is a \mathbb{B} -elliptic cone differential operator with conormal symbol given by

$$\sigma_M(\Delta)(\lambda) = \lambda^2 - (n-1)\lambda + \Delta_{\mathfrak{h}(0)}.$$

Therefore, if $\sigma(\Delta_{\mathfrak{h}(0)}) = \{\lambda_i\}_{i \in \mathbb{N}}$ is the spectrum of $\Delta_{\mathfrak{h}(0)}$, then $(\sigma_M(\Delta)(\lambda))^{-1}$ is a meromorphic in $\lambda \in \mathbb{C}$ family of pseudodifferential operators with values in $\mathcal{L}(H_p^s(\partial\mathbb{B}), H_p^{s+2}(\partial\mathbb{B}))$, $s \in \mathbb{R}$, $p \in (1, \infty)$, and poles that coincide with the set

$$\left\{ \frac{n-1}{2} \pm \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_i} \right\}_{i \in \mathbb{N}}.$$

If $\gamma \in [\frac{n-3}{2}, \frac{n+1}{2})$, then zero is always a pole of $\sigma_M(\Delta)(\lambda)$ and the constant functions on \mathbb{B} , denoted by \mathbb{C} and regarded as a subspace of $\mathcal{E}_{\Delta,\gamma}$, are contained in the maximal domain of Δ in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, $s \in \mathbb{R}$, $p \in (1, \infty)$. Such a realization can satisfy the property of maximal L^q -regularity, as it can be seen from the following result.

Theorem 4.1. *Let $s \geq 0$, $p \in (1, \infty)$ and the weight γ be chosen as*

$$(4.36) \quad \frac{n-3}{2} < \gamma < \min \left\{ -1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n+1}{2} \right\},$$

where λ_1 is the greatest non-zero eigenvalue of the boundary Laplacian $\Delta_{\mathfrak{h}(0)}$. Consider the closed extension $\underline{\Delta}_s$ of Δ in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ with domain

$$(4.37) \quad \mathcal{D}(\underline{\Delta}_s) = \mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}.$$

Then, for any $\theta \in [0, \pi)$ there exists some $c > 0$ such that $c - \underline{\Delta}_s$ is R -sectorial of angle θ .

Proof. The theorem follows by [25, Theorem 5.6]. However, we give here a direct proof inspired by the proof of [26, Theorem 6.1], which we follow.

Recall that $\mathbb{B} = (\mathcal{B}, \mathfrak{g})$ and consider another Riemannian metric \mathfrak{g}_0 on \mathcal{B} such that

$$\mathfrak{g}_0|_{[0,\xi) \times \partial\mathcal{B}} = dx^2 + x^2\mathfrak{h}(0)$$

and $\mathfrak{g}_0 = \mathfrak{g}$ on $\mathcal{B} \setminus \{[0, 1) \times \partial\mathcal{B}\}$, for some fixed $\xi \in (0, 1)$. Denote $\mathbb{B}_0 = (\mathcal{B}, \mathfrak{g}_0)$ and by $\Delta_{\mathfrak{g}_0}$ the associated Laplacian. Note that Δ and $\Delta_{\mathfrak{g}_0}$ have the same conormal symbol. Let $\underline{\Delta}_{\mathfrak{g}_0,s}$ be the realization of $\Delta_{\mathfrak{g}_0}$ in $\mathcal{H}_p^{s,\gamma}(\mathbb{B}_0)$ with domain $\mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}_0) \oplus \mathbb{C}$. Finally, fix the Banach couple $X_0^s = \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $X_1^s = \mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}$.

Assume first that $s = 0$. Let $\underline{\Delta}_0$ be the realization of $\Delta_{\mathfrak{g}_0}$ in X_0^0 with domain X_1^0 . By the choice of the domain of $\underline{\Delta}_{\mathfrak{g}_0,0}$, from [27, Theorem 5.6] and [27, Theorem 5.7] we have that $\underline{\Delta}_{\mathfrak{g}_0,0}$ satisfies the ellipticity condition (E1), (E2) and (E3) of [27, Section 3.2]. Hence, since $\underline{\Delta}_{\mathfrak{g}_0,0}$ and $\underline{\Delta}_0$ have the same model cone operator, we deduce that (E1), (E2) and (E3) are also fulfilled by $\underline{\Delta}_0$. Therefore, by [27, Theorem 4.3], for any $\phi \in (0, \pi)$ there exists some $c_\phi > 0$ such that $c_\phi - \underline{\Delta}_0 \in \mathcal{BIP}(\phi)$. As

a consequence, since X_0^0 is UMD, by [4, Theorem 4] there exists some $c > 0$ such that $c - \underline{A}_0$ is R -sectorial of angle θ .

Take a cut-off function ω on \mathcal{B} with values in $[0, 1]$ satisfying

$$\omega = \begin{cases} 1 & \text{in } [0, \sigma] \times \partial\mathcal{B} \\ 0 & \text{in } \mathcal{B} \setminus \{[0, 2\sigma] \times \partial\mathcal{B}\}, \end{cases}$$

with $\sigma \in (0, \frac{1}{2})$, and consider the operator $\underline{A}_1 = \omega \underline{\Delta}_0 + (1 - \omega) \underline{A}_0$ with domain X_1^0 in X_0^0 . By taking σ sufficiently small, we can make the $\mathcal{L}(X_1^0, X_0^0)$ -norm of $\omega(\underline{\Delta}_0 - \underline{A}_0)$ arbitrary small. Hence, by the relation $\underline{A}_1 = \underline{A}_0 + \omega(\underline{\Delta}_0 - \underline{A}_0)$ and the perturbation result [13, Theorem 1], there exists some $\sigma \in (0, \frac{1}{2})$ such that the operator $c - \underline{A}_1$ is R -sectorial of angle θ .

Endow \mathcal{B} with a metric \mathfrak{g}_2 such that

$$\mathfrak{g}_2|_{[0, \eta] \times \partial\mathcal{B}} = dx^2 + x^2 \mathfrak{h}(0)$$

and $\mathfrak{g}_2 = \mathfrak{g}$ on $\mathcal{B} \setminus \{[0, 2\eta] \times \partial\mathcal{B}\}$, where $\eta \in (0, \frac{\sigma}{2})$. Denote by $\Delta_{\mathfrak{g}_2}$ the Laplacian associated to $(\mathcal{B}, \mathfrak{g}_2)$ and by \underline{A}_2 its realization in X_0^0 with domain X_1^0 . Similarly to \underline{A}_0 , there exists some $c > 0$ such that $c - \underline{A}_2$ is R -sectorial of angle θ .

Take a smooth function ϕ_1 on \mathcal{B} such that $\text{supp}(\phi_1) \subset [0, \sigma] \times \partial\mathcal{B}$ and $\phi_1 = 1$ in $[0, 2\eta + \delta] \times \partial\mathcal{B}$, for some $\delta \in (0, \sigma - 2\eta)$. Assume that ϕ_1 depends only on x and let $\phi_2 = 1 - \phi_1$. Further, let ψ_1, ψ_2 be two smooth functions on \mathcal{B} such that $\text{supp}(\psi_1) \subset [0, \sigma] \times \partial\mathcal{B}$, $\text{supp}(\psi_2) \subset \mathcal{B} \setminus \{[0, 2\eta] \times \partial\mathcal{B}\}$, $\psi_1 = 1$ on $\text{supp}(\phi_1)$ and $\psi_2 = 1$ on $\text{supp}(\phi_2)$. Then, for $f \in X_1^0$, $g \in X_0^0$ and $\lambda \in \mathbb{C}$ by multiplying

$$\lambda f - \underline{\Delta}_0 f = g$$

with ϕ_i , $i \in \{1, 2\}$, we have that

$$\lambda \phi_i f - \underline{\Delta}_0(\phi_i f) = \phi_i g - [\underline{\Delta}_0, \phi_i] f.$$

By restricting λ and applying the resolvent of \underline{A}_i to the previous equation we find that

$$\phi_i f = (\lambda - \underline{A}_i)^{-1} \phi_i g - (\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i] f.$$

By multiplying with ψ_i , the above equation becomes

$$\phi_i f = \psi_i (\lambda - \underline{A}_i)^{-1} \phi_i g - \psi_i (\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i] f,$$

where by summing up we obtain

$$(4.38) \quad f = \sum_{i=1}^2 \psi_i (\lambda - \underline{A}_i)^{-1} \phi_i g - \sum_{i=1}^2 \psi_i (\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i] f.$$

For the first order cone differential operator $[\underline{\Delta}_0, \phi_i]$, by [26, Lemma 5.2], [1, (I.2.5.2)] and [1, (I.2.9.6)] we have that

$$[\underline{\Delta}_0, \phi_i] \in \mathcal{L}(X_1^0, \mathcal{H}_p^{1, \gamma+1}(\mathbb{B})) \subset \mathcal{L}(X_1^0, \mathcal{D}((c_0 - \underline{A}_i)^\nu)), \quad i \in \{1, 2\},$$

for some $\nu \in (0, \frac{1}{2})$ and sufficiently large $c_0 > 0$. Therefore, by writing

$$(\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i] = (\lambda - \underline{A}_i)^{-1} (c_0 - \underline{A}_i)^{-\nu} (c_0 - \underline{A}_i)^\nu [\underline{\Delta}_0, \phi_i]$$

and applying [26, Corollary 2.4], we obtain that

$$\|(c_0 - \underline{A}_i)(\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i]\|_{\mathcal{L}(X_1^0, X_0^0)} \rightarrow 0 \quad \text{as } |\lambda| \rightarrow \infty, \quad |\arg \lambda| \leq \theta.$$

Hence, since multiplication by ψ_i induces a bounded map in X_1^0 , by (4.38), $\lambda - \underline{\Delta}_0$ is left invertible when $\lambda \in \{\mu \in \mathbb{C} \mid |\arg \mu| \leq \theta\}$ and $|\lambda|$ is sufficiently large. Further, since $[\underline{\Delta}_0, \phi_i]$ is of lower order, by [26, Lemma 5.2], [1, (I.2.5.2)], [1, (I.2.9.6)] and [26, Corollary 2.4], we have that

(4.39)

$$[\underline{\Delta}_0, \phi_i](\lambda + c - \underline{A}_j)^{-1} = [\underline{\Delta}_0, \phi_i](c_0 - \underline{A}_j)^{-\tau} (c_0 - \underline{A}_j)^\tau (\lambda + c - \underline{A}_j)^{-1} \in \mathcal{L}(X_0^0), \quad i, j \in \{1, 2\},$$

with some $\tau \in (\frac{1}{2}, 1)$, and for any $\varepsilon > 0$ there exists some $c > 0$ such that

$$(4.40) \quad \|[\underline{\Delta}_0, \phi_i](\lambda + c - \underline{A}_j)^{-1}\|_{\mathcal{L}(X_0^0)} < \varepsilon, \quad \forall i, j \in \{1, 2\},$$

uniformly in $\lambda \in \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}$. Therefore, this left inverse, which we denote by S_λ , belongs to $\mathcal{L}(X_0^0, X_1^0)$ and its $\mathcal{L}(X_0^0, X_1^0)$ -norm is uniformly bounded in $\lambda \in \{c + \mu \in \mathbb{C} \mid |\arg \mu| \leq \theta\}$.

Next, we apply $\lambda - \underline{\Delta}_0$ on (4.38) to obtain

$$\begin{aligned} (\lambda - \underline{\Delta}_0)S_\lambda &= (\lambda - \underline{\Delta}_0) \sum_{i=1}^2 \psi_i(\lambda - \underline{A}_i)^{-1}(\phi_i - [\underline{\Delta}_0, \phi_i]S_\lambda) \\ &= \sum_{i=1}^2 \psi_i(\phi_i - [\underline{\Delta}_0, \phi_i]S_\lambda) - \sum_{i=1}^2 [\underline{\Delta}_0, \psi_i](\lambda - \underline{A}_i)^{-1}(\phi_i - [\underline{\Delta}_0, \phi_i]S_\lambda). \end{aligned}$$

Since $\sum_{i=1}^2 \phi_i = 1$ and $\sum_{i=1}^2 [\underline{\Delta}_0, \phi_i] = 0$, we find that

$$(4.41) \quad (\lambda - \underline{\Delta}_0)S_\lambda = I - \sum_{i=1}^2 [\underline{\Delta}_0, \psi_i](\lambda - \underline{A}_i)^{-1}(\phi_i - [\underline{\Delta}_0, \phi_i]S_\lambda).$$

In the above equation we write

$$(\lambda - \underline{A}_i)^{-1} = (c_0 - \underline{A}_i)^{-\tau} (c_0 - \underline{A}_i)^\tau (\lambda - \underline{A}_i)^{-1},$$

so that similarly to (4.39), the operator $[\underline{\Delta}_0, \psi_i](c_0 - \underline{A}_i)^{-\tau}$ is bounded in X_0^0 . Then, by [26, Corollary 2.4], we can make the $\mathcal{L}(X_0^0)$ -norm of the last term in (4.41) arbitrary small by taking $|\lambda|$ sufficiently large. Hence, $\lambda - \underline{\Delta}_0$ has also a right inverse for large $|\lambda|$ such that $\lambda \in \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}$.

Denote

$$R(\lambda) = \sum_{i=1}^2 \psi_i(\lambda - \underline{A}_i)^{-1} \phi_i \quad \text{and} \quad Q(\lambda) = \sum_{i=1}^2 \psi_i(\lambda - \underline{A}_i)^{-1} [\underline{\Delta}_0, \phi_i].$$

From (4.38), for large λ in $\{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}$ by Neumann series we obtain that

$$(4.42) \quad (\lambda - \underline{\Delta}_0)^{-1} = \sum_{k=0}^{\infty} (-1)^k Q^k(\lambda) R(\lambda).$$

Take any $\lambda_1, \dots, \lambda_N \in \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}$, $x_1, \dots, x_N \in X_0^0$, $N \in \mathbb{N}$, and denote by ϵ_ρ the ρ -th Rademacher function. We use (4.42) to show R -sectoriality for $c - \underline{\Delta}_0$, with $c > 0$ sufficiently large. We start by splitting out the term for $k = 0$ as follows

$$\begin{aligned} &\left\| \sum_{\rho=1}^N \epsilon_\rho \lambda_\rho (\lambda_\rho + c - \underline{\Delta}_0)^{-1} x_\rho \right\|_{L^2(0,1; X_0^0)} \\ &\leq \left\| \sum_{\rho=1}^N \epsilon_\rho \lambda_\rho \left(\sum_{i=1}^2 \psi_i(\lambda_\rho + c - \underline{A}_i)^{-1} \phi_i \right) x_\rho \right\|_{L^2(0,1; X_0^0)} \\ (4.43) \quad &+ \sum_{k=1}^{\infty} \left\| \sum_{\rho=1}^N \epsilon_\rho \lambda_\rho Q^k(\lambda_\rho + c) \left(\sum_{i=1}^2 \psi_i(\lambda_\rho + c - \underline{A}_i)^{-1} \phi_i \right) x_\rho \right\|_{L^2(0,1; X_0^0)}. \end{aligned}$$

For the first term on the right hand side of (4.43) we have that

$$\begin{aligned}
& \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} \left(\sum_{i=1}^2 \psi_i (\lambda_{\rho} + c - \underline{A}_i)^{-1} \phi_i \right) x_{\rho} \right\|_{L^2(0,1;X_0^0)} \\
& \leq \left\| \sum_{i=1}^2 \psi_i \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{A}_i)^{-1} \phi_i x_{\rho} \right\|_{L^2(0,1;X_0^0)} \\
& \leq \sum_{i=1}^2 \|\psi_i\|_{\mathcal{L}(X_0^0)} \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{A}_i)^{-1} \phi_i x_{\rho} \right\|_{L^2(0,1;X_0^0)} \\
& \leq C_1 \max_{i \in \{1,2\}} \left\| \sum_{\rho=1}^N \phi_i \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1;X_0^0)} \\
(4.44) \quad & \leq C_2 \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1;X_0^0)},
\end{aligned}$$

for some constants $C_1, C_2 > 0$, where we have used the R -sectoriality of $c - \underline{A}_i$, $i \in \{1, 2\}$.

Concerning the second term on the right hand side of (4.43), for each k and ρ we expand the term

$$\lambda_{\rho} Q^k (\lambda_{\rho} + c) \sum_{i=1}^2 \psi_i (\lambda_{\rho} + c - \underline{A}_i)^{-1} \phi_i$$

into a sum of a 2^{k+1} products of $k+1$ factors. The first factor equals to

$$\psi_i \lambda_{\rho} (\lambda_{\rho} + c - \underline{A}_i)^{-1}, \quad i \in \{1, 2\},$$

and the rest k are of the form

$$[\underline{\Delta}_0, \phi_j] \psi_i (\lambda_{\rho} + c - \underline{A}_i)^{-1} = [\underline{\Delta}_0, \phi_j] \psi_i (c - \underline{A}_i)^{-1} (c - \underline{A}_i) (\lambda_{\rho} + c - \underline{A}_i)^{-1}, \quad i, j \in \{1, 2\}.$$

Similarly to (4.40), we can choose $c > 0$ sufficiently large so that

$$\|[\underline{\Delta}_0, \phi_j] \psi_i (c - \underline{A}_i)^{-1}\|_{\mathcal{L}(X_0^0)} < \varepsilon \quad \text{for all } i, j \in \{1, 2\}.$$

Therefore, by using successively the R -sectoriality of $c - \underline{A}_i$ together with [26, Lemma 2.6] we estimate the second term on the right hand side of (4.43) by

$$\begin{aligned}
& \sum_{k=1}^{\infty} \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} Q^k (\lambda_{\rho} + c) \left(\sum_{i=1}^2 \psi_i (\lambda_{\rho} + c - \underline{A}_i)^{-1} \phi_i \right) x_{\rho} \right\|_{L^2(0,1;X_0^0)} \\
(4.45) \quad & \leq C_3 \sum_{k=1}^{\infty} (C\varepsilon)^k \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1;X_0^0)},
\end{aligned}$$

with some constants $C_3, C > 0$ and $\varepsilon > 0$ small enough. Hence, the R -sectoriality of $c - \underline{\Delta}_0$ follows by (4.43), (4.44) and (4.45).

Consider now the case of $s > 0$. Take $\lambda \in \rho(\underline{\Delta}_0)$, $w \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and let $f = (\lambda - \underline{\Delta}_0)^{-1} w$. Since $(\lambda - \underline{\Delta}_0)f = w$, we have that $f \in \mathcal{D}(\underline{\Delta}_{s,\max}) \cap (\mathcal{H}_p^{2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C})$, and by the domain description (3.33) we conclude that $f \in \mathcal{H}_p^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}$. Therefore,

$$(\lambda - \underline{\Delta}_s)(\lambda - \underline{\Delta}_0)^{-1} = I \quad \text{in } X_0^s.$$

Furthermore,

$$(\lambda - \underline{\Delta}_0)^{-1}(\lambda - \underline{\Delta}_s) = I \quad \text{in } X_1^s.$$

So we conclude that $\rho(\underline{\Delta}_0) \subseteq \rho(\underline{\Delta}_s)$ and that the resolvent of $\underline{\Delta}_s$ for $s > 0$ is the restriction of the resolvent of $\underline{\Delta}_0$ to X_0^s .

We first treat the case of $s \in \mathbb{N}$ by induction. Take an open covering $U_i \subseteq \mathcal{B}$, $i \in \{1, \dots, K\}$, $K \in \mathbb{N}$, and let $\{\omega_i\}_{i \in \{1, \dots, K\}}$ be a subordinated partition of unity. Take local coordinates y_0, \dots, y_n on the support of ω_i , such that when we are in the collar neighborhood of the boundary $[0, 1) \times \partial\mathcal{B}$, we choose $y_0 = x$ and replace the derivative ∂_{y_0} with the Mellin derivative $x\partial_x$. Finally, we denote by M_{ω_i} the multiplication operator by ω_i . We have that

$$\begin{aligned}
& \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_{s+1})^{-1} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))} \\
&= \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))} \\
&\leq \sum_{|a| \leq 1} \left\| \sum_{i=1}^K \sum_{\rho=1}^N \partial_y^a M_{\omega_i} \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\leq \sum_{|a| \leq 1} \left\| \sum_{i=1}^K \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} \partial_y^a M_{\omega_i} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\quad + \sum_{|a| \leq 1} \left\| \sum_{i=1}^K \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} [\partial_y^a M_{\omega_i}, (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1}] x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&= \sum_{|a| \leq 1} \left\| \sum_{\rho=1}^N \sum_{i=1}^K \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} \partial_y^a M_{\omega_i} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\quad + \sum_{|a| \leq 1} \left\| \sum_{\rho=1}^N \sum_{i=1}^K \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} [\partial_y^a M_{\omega_i}, c - \underline{\Delta}_s] \right. \\
&\quad \times (c - \underline{\Delta}_s)^{-1} (c - \underline{\Delta}_s) (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} x_{\rho} \left. \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))}.
\end{aligned} \tag{4.46}$$

Each commutator $[\partial_y^a M_{\omega_i}, c - \underline{\Delta}_s]$ is a second order cone differential operator and therefore the terms $[\partial_y^a M_{\omega_i}, c - \underline{\Delta}_s](c - \underline{\Delta}_s)^{-1}$ belong to $\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}))$. Hence, by the R -sectoriality of $c - \underline{\Delta}_s$ in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, (4.46) implies that

$$\begin{aligned}
& \left\| \sum_{\rho=1}^N \epsilon_{\rho} \lambda_{\rho} (\lambda_{\rho} + c - \underline{\Delta}_{s+1})^{-1} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))} \\
&\leq C_4 \sum_{|a| \leq 1} \left\| \sum_{\rho=1}^N \sum_{i=1}^K \partial_y^a M_{\omega_i} \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\quad + C_4 \sum_{|a| \leq 1} \left\| \sum_{\rho=1}^N \sum_{i=1}^K [\partial_y^a M_{\omega_i}, c - \underline{\Delta}_s] (c - \underline{\Delta}_s)^{-1} (c - \underline{\Delta}_s) (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\leq C_5 \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))} + C_5 \max_{i,a} \left\| [\partial_y^a M_{\omega_i}, c - \underline{\Delta}_s] (c - \underline{\Delta}_s)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\quad \times \left\| \sum_{\rho=1}^N (c - \underline{\Delta}_s) (\lambda_{\rho} + c - \underline{\Delta}_s)^{-1} \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\leq C_5 \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))} + C_6 \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s,\gamma}(\mathbb{B}))} \\
&\leq C_7 \left\| \sum_{\rho=1}^N \epsilon_{\rho} x_{\rho} \right\|_{L^2(0,1; \mathcal{H}_p^{s+1,\gamma}(\mathbb{B}))},
\end{aligned}$$

for certain positive constants C_4, C_5, C_6 and C_7 .

Finally, the case where $s > 0$ is not an integer, follows by interpolation by Lemma 3.4 and [11, Theorem 3.19]. \square

The domain of the bi-Laplacian associated to the Laplacian $\underline{\Delta}_s$ from (4.36)-(4.37) is of particular interest for the study of the Swift-Hohenberg equation. The conormal symbol $\sigma_M(\Delta^2)(\lambda)$ of Δ^2 , by [27, (2.13)], can be expressed as

$$\sigma_M(\Delta^2)(\lambda) = \sigma_M(\Delta)(\lambda)\sigma_M(\Delta)(\lambda + 2).$$

The description of $\mathcal{D}(\underline{\Delta}_s^2)$ can be obtained by the poles in $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \in [\frac{n+1}{2} - \gamma - 4, \frac{n+1}{2} - \gamma - 2]\}$ of the families of symbols defined in (3.34), although it is also given in [25, Section 5.3]. Therefore, we summarize as follows.

The domain of the bi-Laplacian. *The domain of the bi-Laplacian $\underline{\Delta}_s^2$ associated to the Laplacian $\underline{\Delta}_s$ from (4.36)-(4.37), defined as usual by*

$$(4.47) \quad \mathcal{D}(\underline{\Delta}_s^2) = \{u \in \mathcal{D}(\underline{\Delta}_s) \mid \underline{\Delta}_s u \in \mathcal{D}(\underline{\Delta}_s)\},$$

has the following form

$$(4.48) \quad \mathcal{D}(\underline{\Delta}_s^2) = \mathcal{D}(\underline{\Delta}_{\min,s}^2) \oplus \mathbb{C} \oplus \bigoplus_{\rho} \mathcal{F}_{\rho}.$$

Here for the minimal domain we have that $\mathcal{D}(\underline{\Delta}_{\min,s}^2) \subset \mathcal{H}_p^{s+4, \gamma+4-\varepsilon}(\mathbb{B})$, for any $\varepsilon > 0$, and the asymptotics spaces \mathcal{F}_{ρ} are s -independent finite dimensional spaces consisting of linear combinations of smooth functions on \mathcal{B} such that on $(0, 1) \times \partial\mathcal{B}$ they take the form

$$c(y)x^{-\rho} \log^k(x), \quad k \in \{0, 1\},$$

where $c \in C^\infty(\partial\mathcal{B})$ and the exponents ρ lie in the strip $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) \in [\frac{n+1}{2} - \gamma - 4, \frac{n+1}{2} - \gamma - 2]\}$. However, by (4.47) we conclude that

$$\bigoplus_{\rho} \mathcal{F}_{\rho} \subset \mathcal{H}_p^{s+2, \gamma+2}(\mathbb{B}).$$

We end this section by showing that the above bi-Laplacian satisfies the property of maximal L^q -regularity.

Lemma 4.2. *Let $s \geq 0$, $p \in (1, \infty)$, γ be chosen as in (4.36) and $\underline{\Delta}_s$ be the Laplacian (4.37). Then, for any $\theta \in [0, \pi)$ there exists some $c > 0$ such that the operator $\underline{\Delta}_s^2 + c$ is R -sectorial of angle θ .*

Proof. Denote $X_0^s = \mathcal{H}_p^{s, \gamma}(\mathbb{B})$ and let $\lambda_1, \dots, \lambda_N \in \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}$, $x_1, \dots, x_N \in X_0^s$, $N \in \mathbb{N}$. By Theorem 4.1, for any $\phi \in [0, \pi)$ there exists some c_ϕ depending on ϕ such that $c_\phi - \underline{\Delta}_s$ is R -sectorial of angle ϕ . Then, by taking $\phi = \frac{\theta + \pi}{2}$ we have that

$$\begin{aligned} & \left\| \sum_{i=1}^N \epsilon_i \lambda_i ((c_\phi - \underline{\Delta}_s)^2 + \lambda_i)^{-1} x_i \right\|_{L^q(0,1; X_0^s)} \\ &= \left\| \sum_{i=1}^N \epsilon_i \lambda_i (c_\phi - \underline{\Delta}_s - i\sqrt{\lambda_i})^{-1} (c_\phi - \underline{\Delta}_s + i\sqrt{\lambda_i})^{-1} x_i \right\|_{L^q(0,1; X_0^s)} \\ &= \left\| \sum_{i=1}^N \epsilon_i \frac{\lambda_i}{2i\sqrt{\lambda_i}} ((c_\phi - \underline{\Delta}_s - i\sqrt{\lambda_i})^{-1} - (c_\phi - \underline{\Delta}_s + i\sqrt{\lambda_i})^{-1}) x_i \right\|_{L^q(0,1; X_0^s)} \\ &\leq \left\| \sum_{i=1}^N \epsilon_i \frac{i\sqrt{\lambda_i}}{2} (c_\phi - \underline{\Delta}_s - i\sqrt{\lambda_i})^{-1} x_i \right\|_{L^q(0,1; X_0^s)} \\ &\quad + \left\| \sum_{i=1}^N \epsilon_i \frac{i\sqrt{\lambda_i}}{2} (c_\phi - \underline{\Delta}_s + i\sqrt{\lambda_i})^{-1} x_i \right\|_{L^q(0,1; X_0^s)} \\ (4.49) \quad &\leq C \left\| \sum_{i=1}^N \epsilon_i x_i \right\|_{L^q(0,1; X_0^s)}, \end{aligned}$$

for suitable constant $C > 0$. Therefore, $(c_\phi - \underline{\Delta}_s)^2$ is R -sectorial of angle θ . Furthermore, for any $c > 0$, by [26, Lemma 2.6] we have that $(c_\phi - \underline{\Delta}_s)^2 + c$ is also R -sectorial of angle θ and its R -sectorial bound is uniformly bounded in c .

Next, we write

$$(4.50) \quad \underline{\Delta}_s^2 + c = (c_\phi - \underline{\Delta}_s)^2 + c - c_\phi^2 + 2c_\phi \underline{\Delta}_s.$$

We have that

$$\begin{aligned} & \| (c_\phi^2 - 2c_\phi \underline{\Delta}_s) ((c_\phi - \underline{\Delta}_s)^2 + c)^{-1} \|_{\mathcal{L}(X_0^s)} \\ & \leq \| (c_\phi^2 - 2c_\phi \underline{\Delta}_s) (c_\phi - \underline{\Delta}_s)^{-1} \|_{\mathcal{L}(X_0^s)} \| (c_\phi - \underline{\Delta}_s) ((c_\phi - \underline{\Delta}_s)^2 + c)^{-1} \|_{\mathcal{L}(X_0^s)}, \end{aligned}$$

and therefore

$$\| (c_\phi^2 - 2c_\phi \underline{\Delta}_s) ((c_\phi - \underline{\Delta}_s)^2 + c)^{-1} \|_{\mathcal{L}(X_0^s)} \rightarrow 0 \quad \text{as } c \rightarrow \infty,$$

due to [26, Corollary 2.4]. Then, the result follows by (4.50) and the perturbation result [13, Theorem 1]. \square

5. MAXIMAL L^q -REGULARITY AND LONG TIME EXISTENCE OF SOLUTIONS FOR THE SWIFT-HOHENBERG EQUATION ON MANIFOLDS WITH CONICAL SINGULARITIES

In this section we consider the problem (1.1)-(1.2) on a manifold \mathbb{B} with warped cones, i.e. with singularities described by (3.32). We show existence, uniqueness and maximal L^q -regularity of the short time solution in arbitrary high order Mellin-Sobolev spaces. We also provide information concerning the asymptotic behavior of the solution close to the conical tip. Furthermore, under certain assumption we show that the above solution exists for all times. In order to treat the non-linearity, we make some further restriction on the data.

Choice of the data. Let λ_1 be the greatest non-zero eigenvalue of the boundary Laplacian $\Delta_{\mathfrak{h}(0)}$ induced by the metric $\mathfrak{h}(x)$ from (3.32) frozen at $x = 0$. Assume that $p, q \in (1, \infty)$ are sufficiently large such that

$$(5.51) \quad \frac{2}{q} < \min \left\{ -1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n+1}{2} \right\} - \frac{n-3}{2} \quad \text{and} \quad \frac{2}{q} + \frac{n+1}{p} < 2,$$

and choose the weight γ as follows

$$(5.52) \quad \frac{n-3}{2} + \frac{2}{q} < \gamma < \min \left\{ -1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}, \frac{n+1}{2} \right\}.$$

With the data chosen as above, we consider the Banach couple $X_0^s = \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $X_2^s = \mathcal{D}(\underline{\Delta}_s^2)$, where $\underline{\Delta}_s$ is the closed extension of Δ given by (4.37). Furthermore, we denote

$$X_{\frac{1}{q},q}^s = (X_2^s, X_0^s)_{\frac{1}{q},q}.$$

We start with the short time existence and space asymptotics of the solution result, that is based on the maximal L^q -regularity property of the Laplacian on arbitrary high order Mellin-Sobolev spaces.

Theorem 5.1 (Short time solution). *Let $s \geq 0$ and p, q, γ be chosen as in (5.51)-(5.52). Then, for any*

$$u_0 \in (\mathcal{D}(\underline{\Delta}_s^2), \mathcal{H}_p^{s,\gamma}(\mathbb{B}))_{\frac{1}{q},q}$$

there exists a $T > 0$ and a unique

$$u \in W^{1,q}(0, T; \mathcal{H}_p^{s,\gamma}(\mathbb{B})) \cap L^q(0, T; \mathcal{D}(\underline{\Delta}_s^2))$$

solving the problem (1.1)-(1.2) on \mathbb{B} , where the bi-Laplacian domain is described in (4.48).

Furthermore, for each $\delta \in (0, T)$ we have that

$$u \in W^{1,\infty}(\delta, T; \mathcal{H}_p^{s,\gamma}(\mathbb{B})) \cap L^\infty(\delta, T; \mathcal{D}(\underline{\Delta}_s^2))$$

with $u'(t) \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$ and $u(t) \in \mathcal{D}(\underline{\Delta}_s^2)$ for each $t \in [\delta, T]$.

Proof. We will apply the theorem of Clément and Li to the problem (1.1)-(1.2) with $A = (\underline{\Delta}_s + 1)^2$ and $G(t, u) = F(t, u)$. For any $\theta \in (\frac{\pi}{2}, \pi)$, by Theorem 4.1, there exists some $c_0 > 0$ such that $c_0 - \underline{\Delta}_s$ is sectorial of angle θ . Take $c_1 = \max\{0, c_0^2 \tan^2(\theta) - 1\}$, so that $c > c_1$ implies

$$-c_0 \pm i\sqrt{1+c} \in \{\mu \in \mathbb{C} \mid |\arg(\mu)| \leq \theta\}.$$

For $c > c_1$ sufficiently large, by the sectoriality of $c_0 - \underline{\Delta}_s$, we can make the $\mathcal{L}(X_0^s)$ -norm of

$$2\underline{\Delta}_s(\underline{\Delta}_s^2 + 1 + c)^{-1} = 2\underline{\Delta}_s(c_0 - \underline{\Delta}_s - c_0 + i\sqrt{1+c})^{-1}(c_0 - \underline{\Delta}_s - c_0 - i\sqrt{1+c})^{-1}$$

arbitrary small. Therefore, by Lemma 4.2, [26, Lemma 2.6] and [13, Theorem 1], there exists some $c > 0$ such that the operator $A + c = \underline{\Delta}_s^2 + 2\underline{\Delta}_s + 1 + c$ with domain X_2^s in X_0^s is R -sectorial of angle θ . Hence, the condition (H1) of Theorem 2.11 is satisfied.

Concerning the real interpolation space $(X_2^s, X_0^s)_{\frac{1}{q}, q}$, from [26, Lemma 5.2] we estimate

$$(5.53) \quad (X_0^s, X_2^s)_{1-\frac{1}{q}, q} \hookrightarrow (X_0^s, \mathcal{D}(\underline{\Delta}_s))_{1-\frac{1}{q}, q} \hookrightarrow \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}, \quad \text{for all } \varepsilon > 0.$$

Let U_{u_0} be a bounded open neighborhood of $u_0 \in X_{\frac{1}{q}, q}^s$ and let $u_1, u_2 \in U_{u_0}$. Further, take $T_0 > 0$ and let $t_1, t_2 \in [0, T_0]$. We have that

$$(5.54) \quad F(t, u) = \sum_{k=0}^m \alpha_k(t) u^k,$$

for some $m \in \mathbb{N}$ and a collection of Lipschitz on \mathbb{R} functions $\{\alpha_k\}_{k \in \{0, \dots, m\}}$. If we denote by L_{α_k} the Lipschitz bound of α_k , by Lemma 3.3 and (5.53) we estimate

$$\begin{aligned} & \|F(t_1, u_1) - F(t_2, u_2)\|_{X_0^s} \\ & \leq \left\| \sum_{k=0}^m \alpha_k(t_1)(u_1^k - u_2^k) \right\|_{X_0^s} + \left\| \sum_{k=0}^m (\alpha_k(t_1) - \alpha_k(t_2)) u_2^k \right\|_{X_0^s} \\ & \leq \left(\max_{k \in \{1, \dots, m\}} \sup_{\tau \in [0, T_0]} |\alpha_k(\tau)| \right) \sum_{k=1}^m \|u_1^k - u_2^k\|_{X_0^s} + |t_1 - t_2| \left(\max_{k \in \{0, \dots, m\}} L_{\alpha_k} \right) \sum_{k=0}^m \|u_2^k\|_{X_0^s} \\ & \leq C_1 \left(\max_{k \in \{1, \dots, m\}} \sup_{\tau \in [0, T_0]} |\alpha_k(\tau)| \right) \left(\sum_{k=1}^m \sum_{i=0}^{k-1} \|u_1\|_{X_{\frac{1}{q}, q}^s}^{k-1-i} \|u_2\|_{X_{\frac{1}{q}, q}^s}^i \right) \|u_1 - u_2\|_{X_0^s} \\ & \quad + C_1 |t_1 - t_2| \left(\max_{k \in \{0, \dots, m\}} L_{\alpha_k} \right) \sum_{k=0}^m \|u_2\|_{X_{\frac{1}{q}, q}^s}^{k-1} \|u_2\|_{X_0^s} \\ & \leq C_2 \left(\max_{k \in \{1, \dots, m\}} \sup_{\tau \in [0, T_0]} |\alpha_k(\tau)| \right) \left(\sum_{k=1}^m \sum_{i=0}^{k-1} \|u_1\|_{X_{\frac{1}{q}, q}^s}^{k-1-i} \|u_2\|_{X_{\frac{1}{q}, q}^s}^i \right) \|u_1 - u_2\|_{X_{\frac{1}{q}, q}^s} \\ & \quad + C_2 |t_1 - t_2| \left(\max_{k \in \{0, \dots, m\}} L_{\alpha_k} \right) \sum_{k=0}^m \|u_2\|_{X_{\frac{1}{q}, q}^s}^k \end{aligned} \quad (5.55)$$

for some constants $C_1, C_2 > 0$ depending only on s, p, q and γ . Therefore, the condition (H3) of Theorem 2.11 is also satisfied and hence there exists a $T > 0$ and a unique

$$u \in W^{1,q}(0, T; X_0^s) \cap L^q(0, T; X_2^s)$$

solving the problem (1.1)-(1.2) on \mathbb{B} .

Consider the following linear degenerate parabolic equation

$$(5.56) \quad v'(t) + ((\underline{\Delta}_s + 1)^2 + c)v(t) = e^{-ct} F(t, u(t)), \quad t \in (0, T),$$

$$(5.57) \quad v(0) = u_0.$$

Clearly, the above equation has a solution

$$(5.58) \quad e^{-ct} u \quad \text{in} \quad W^{1,q}(0, T; X_0^s) \cap L^q(0, T; X_2^s).$$

By (2.5), (5.53) and Lemma 3.2 for each $k \in \mathbb{N}$ we have that

$$u^k \in C([0, T]; \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C})$$

for all $\varepsilon > 0$ sufficiently small. Thus, by (5.54) we obtain that

$$(5.59) \quad F(t, u(t)) \in C([0, T]; \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C})$$

for all $\varepsilon > 0$ sufficiently small. Hence, by extending $F(t, u(t))$ to (T, ∞) by zero, from Theorem 2.9 the problem (5.56)-(5.57) has a unique solution $v \in W^{1,q}(0, T; X_0^s) \cap L^q(0, T; X_2^s)$. Therefore, by (5.58) we have that $v = e^{-ct}u$.

Since $c_0 - \underline{\Delta}_s$ is sectorial, by [26, Lemma 5.2], [1, (I.2.5.2)] and [1, (I.2.9.6)] we have that

$$\begin{aligned} & \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C} \\ & \hookrightarrow (X_0^s, \mathcal{D}(c_0 - \underline{\Delta}_s))_{1-\frac{1}{q}-\varepsilon, q} \hookrightarrow (X_0^s, (X_0^s, \mathcal{D}((c_0 - \underline{\Delta}_s)^2))_{\frac{1}{2}-\varepsilon, q})_{1-\frac{1}{q}-\varepsilon, q} \end{aligned}$$

for all $\varepsilon > 0$ sufficiently small. Therefore, by [1, (I.2.5.2)] and reiteration [1, (I.2.8.4)] we obtain that

$$(5.60) \quad \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C} \hookrightarrow (X_0^s, X_2^s)_{\frac{1}{2}-\frac{1}{2q}-4\varepsilon, q}$$

for all $\varepsilon > 0$ sufficiently small. By the above embedding and (5.59) we find that

$$F(t, u(t)) \in C([0, T]; (X_0^s, X_2^s)_{\frac{1}{2}-\frac{1}{2q}-4\varepsilon, q})$$

for any $\varepsilon > 0$ sufficiently small. Then, for any $\delta \in (0, T)$, Theorem 2.9 applied to (5.56)-(5.57) implies that

$$(5.61) \quad u(t) \in X_2^s \quad \text{for each } t \in [\delta, T] \quad \text{and} \quad u \in L^\infty(\delta, T; X_2^s).$$

The rest of the proof follows by (1.1), (5.59) and (5.61). \square

As pointed out in the proof of the above theorem, more precise information about the asymptotic behavior of the solution close to the singularity is obtained by the embedding (2.5) combined with (5.53) and Lemma 3.2. Namely, we have the following.

Corollary 5.2. *Let $s \geq 0$ and p, q, γ be chosen as in (5.51)-(5.52). Then, for the unique solution u of Theorem 5.1 we have that*

$$(5.62) \quad u \in C([0, T]; (X_2^s, X_0^s)_{\frac{1}{q}, q}) \hookrightarrow C([0, T]; \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}) \hookrightarrow C([0, T]; C(\mathbb{B}))$$

for all $\varepsilon > 0$ sufficiently small.

However, for large values of q we have better regularity for the interpolation space, as it can be seen by the following result.

Proposition 5.3. *Let $s \geq 0$ and p, q, γ be chosen as in (5.51)-(5.52) with $q > 2$. Then,*

$$(X_2^s, X_0^s)_{\frac{1}{q}, q} \hookrightarrow \mathcal{H}_p^{s+2, \gamma+2}(\mathbb{B}) \oplus \mathbb{C} \hookrightarrow C(\mathbb{B}).$$

Proof. Let $c_0 > 0$ such that by Theorem 4.1 and Lemma 4.2, both $c_0 - \underline{\Delta}_s$ and $\underline{\Delta}_s^2 + c_0$ are sectorial of angle θ . We have that $X_2^s = \mathcal{D}((c_0 - \underline{\Delta}_s)^2)$ with equivalence of the respective norms. Therefore, by [1, (I.2.5.2)] and [1, (I.2.9.6)] we obtain

$$(X_2^s, X_0^s)_{\frac{1}{q}, q} = (X_0^s, X_2^s)_{1-\frac{1}{q}, q} \hookrightarrow (X_0^s, \mathcal{D}((c_0 - \underline{\Delta}_s)^2))_{1-\frac{1}{q}, q} \hookrightarrow \mathcal{D}(c_0 - \underline{\Delta}_s).$$

Continuity follows by Lemma 3.2. \square

In order to proceed to the existence of the long time solution for the problem (1.1)-(1.2), according to (5.59) we impose certain assumption concerning the term $F(t, u(t))$, under which we can generalize Theorem 5.1 to the following.

Theorem 5.4 (Long time solution). *Let $s \geq 0$, p, q, γ be chosen as in (5.51)-(5.52),*

$$u_0 \in (\mathcal{D}(\underline{\Delta}_s^2), \mathcal{H}_p^{s, \gamma}(\mathbb{B}))_{\frac{1}{q}, q}$$

with $\mathcal{D}(\underline{\Delta}_s^2)$ described in (4.48), and u, T given by Theorem 5.1.

For all such T , let that

$$(5.63) \quad \|F(t, u(t))\|_{L^q(0, T; \mathcal{H}_p^{s, \gamma}(\mathbb{B}))} \leq K(T)$$

for some positive function $K(\cdot) \in C(\mathbb{R})$.

Then, T in Theorem 5.1 can be taken arbitrary large if and only if (5.63) holds.

Proof. As explained in the proof of Theorem 5.1, for any $\theta \in (\frac{\pi}{2}, \pi)$ there exists some $c > 0$ such that the operator $A = (\underline{\Delta}_s + 1)^2 + c$ with domain X_2^s in X_0^s is R -sectorial of angle θ , where $\underline{\Delta}_s$ is the closed extension (4.37). Furthermore, in the proof of Theorem 5.1 it was pointed out that the following linear degenerate parabolic equation

$$(5.64) \quad v'(t) + ((\underline{\Delta}_s + 1)^2 + c)v(t) = e^{-ct}F(t, u(t)), \quad t \in (0, T),$$

$$(5.65) \quad v(0) = u_0$$

has a unique solution

$$v = e^{-ct}u \quad \text{in} \quad W^{1, q}(0, T; X_0^s) \cap L^q(0, T; X_2^s).$$

Assume that (5.63) holds. By extending $F(t, u(t))$ to (T, ∞) by zero, from (5.59) and (2.9) applied to (5.64)-(5.65) we estimate

$$\begin{aligned} & \|u\|_{W^{1, q}(0, T; X_0^s)} + \|u\|_{L^q(0, T; X_2^s)} \\ & \leq (c+1)e^{cT}(\|v\|_{W^{1, q}(0, T; X_0^s)} + \|v\|_{L^q(0, T; X_2^s)}) \\ & \leq (c+1)e^{cT}C_1(\|e^{-ct}F(t, u(t))\|_{L^q(0, T; X_0^s)} + \|u_0\|_{(X_2^s, X_0^s)_{\frac{1}{q}, q}}) \\ (5.66) \quad & \leq (c+1)e^{cT}C_1(K(T) + \|u_0\|_{(X_2^s, X_0^s)_{\frac{1}{q}, q}}) \end{aligned}$$

for some constant C_1 independent of T . Thus, due to the embedding (2.5), there exists some positive function $K_1(\cdot) \in C(\mathbb{R})$ such that

$$(5.67) \quad \|u\|_{C([0, T]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \leq K_1(T).$$

By (5.53), (5.54), (5.60), (5.62) and (5.67) and the Banach algebra property of the space $\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}$ for $\varepsilon > 0$ small enough provided by Lemma 3.2, we obtain that

$$F(t, u(t)) \in L^\infty(0, T; (X_0^s, X_2^s)_{\frac{1}{2}-\frac{1}{2q}-\varepsilon, q})$$

for any $\varepsilon > 0$ sufficiently small and furthermore

$$(5.68) \quad \|F(t, u(t))\|_{L^\infty(0, T; (X_0^s, X_2^s)_{\frac{1}{2}-\frac{1}{2q}-\varepsilon, q})} \leq K_2(T)$$

for some positive function $K_2(\cdot) \in C(\mathbb{R})$. Let us fix $\delta \in (0, T)$. Then, Theorem 2.9 applied to (5.64)-(5.65) implies that

$$(5.69) \quad u(t) \in X_2^s \quad \text{for each} \quad t \in [\delta, T] \quad \text{and} \quad \|u(t)\|_{X_2^s} \leq K_3(T), \quad t \in [\delta, T],$$

for some positive function $K_3(\cdot) \in C(\mathbb{R})$ independent of t .

Let $\tau \in [\delta, T)$ and consider the following linear degenerate parabolic problem,

$$(5.70) \quad h'(t) + (\underline{\Delta}_s + 1)^2 h(t) = F(t, u(\tau)), \quad t > 0,$$

$$(5.71) \quad h(0) = u(\tau).$$

If we let $h(t) = e^{ct}g(t)$, we obtain that

$$(5.72) \quad g'(t) + ((\underline{\Delta}_s + 1)^2 + c)g(t) = e^{-ct}F(t, u(\tau)), \quad t > 0,$$

$$(5.73) \quad g(0) = u(\tau).$$

By Lemma 3.2 and (5.53), for each $\tau \in [\delta, T)$ and $T_1 > 0$ we have that

$$F(t, u(\tau)) \in C([0, T_1]; \mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}) \hookrightarrow L^q(0, T_1; X_0^s)$$

for all $\varepsilon > 0$ sufficiently small. Hence, by Theorem 2.9, there exists a unique $g \in W^{1,q}(0, T_1; X_0^s) \cap L^q(0, T_1; X_2^s)$ solving (5.72)-(5.73). Therefore, the function $w(t) = g(t) - e^{-tA}u(\tau)$ satisfies the equation

$$\begin{aligned} w'(t) + ((\underline{\Delta}_s + 1)^2 + c)w(t) &= e^{-ct}F(t, u(\tau)), \quad t > 0, \\ w(0) &= 0. \end{aligned}$$

Fix $T_2 > 0$ and assume that $T_1 \in (0, T_2]$. By (2.5), (2.9), (5.53) and (5.54) for any $\varepsilon > 0$ small enough we have that

$$\begin{aligned} & \|w(t)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ & \leq C_2 \|w(t)\|_{W^{1,q}(0, T_1; X_0^s) \cap L^q(0, T_1; X_2^s)} \\ & \leq C_3 \|e^{-ct}F(t, u(\tau))\|_{L^q(0, T_1; X_0^s)} \\ & \leq C_3 T_1^{\frac{1}{q}} \left(\max_{k \in \{0, \dots, m\}} \sup_{t \in [0, T_1]} |\alpha_k(t)| \right) \sum_{k=0}^m \|u^k(\tau)\|_{X_0^s} \\ & \leq C_4 T_1^{\frac{1}{q}} \left(\max_{k \in \{0, \dots, m\}} \sup_{t \in [0, T_1]} |\alpha_k(t)| \right) \sum_{k=0}^m \|u^k(\tau)\|_{\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}} \\ & \leq C_5 T_1^{\frac{1}{q}} \left(\max_{k \in \{0, \dots, m\}} \sup_{t \in [0, T_1]} |\alpha_k(t)| \right) \sum_{k=0}^m \|u(\tau)\|_{\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}}^k \\ (5.74) \quad & \leq C_6 T_1^{\frac{1}{q}} \left(\max_{k \in \{0, \dots, m\}} \sup_{t \in [0, T_1]} |\alpha_k(t)| \right) \sum_{k=0}^m \|u(\tau)\|_{(X_2^s, X_0^s)_{\frac{1}{q}, q}}^k \end{aligned}$$

for some positive constants C_2, C_3, C_4, C_5 and C_6 independent of τ, T and T_1 , where we have used the Banach algebra property of $\mathcal{H}_p^{s+2-\frac{2}{q}-\varepsilon, \gamma+2-\frac{2}{q}-\varepsilon}(\mathbb{B}) \oplus \mathbb{C}$ due to Lemma 3.2.

By (5.69), the function $f(t) = u(\tau) - e^{-tA}u(\tau)$ satisfies

$$\begin{aligned} f'(t) + ((\underline{\Delta}_s + 1)^2 + c)f(t) &= ((\underline{\Delta}_s + 1)^2 + c)u(\tau), \quad t > 0, \\ f(0) &= 0. \end{aligned}$$

Hence, by (2.5), (2.9) and (5.69), we obtain that

$$(5.75) \quad \|(e^{-tA} - I)u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \leq K_4(T) T_1^{\frac{1}{q}}$$

for some positive function $K_4(\cdot) \in C(\mathbb{R})$ independent of τ and T_1 .

Therefore, returning to the solution of (5.70)-(5.71) we estimate

$$\begin{aligned} \phi(T_1) &= \|h(t) - u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\leq e^{cT_1} \|g(t) - e^{-ct}u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\leq e^{cT_1} \|w(t)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} + e^{cT_1} \|e^{-tA}u(\tau) - e^{-ct}u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\leq e^{cT_1} \|w(t)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} + e^{cT_1} \|(e^{-tA} - I)u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\quad + e^{cT_1} \|(1 - e^{-ct})u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\leq e^{cT_1} \|w(t)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} + e^{cT_1} \|(e^{-tA} - I)u(\tau)\|_{C([0, T_1]; (X_2^s, X_0^s)_{\frac{1}{q}, q})} \\ &\quad + (e^{cT_1} - 1) \|u(\tau)\|_{(X_2^s, X_0^s)_{\frac{1}{q}, q}}. \end{aligned}$$

Hence, by (5.67), (5.74) and (5.75) we find that $\phi(T_1) \rightarrow 0$ as $T_1 \rightarrow 0$ uniformly in $\tau \in [\delta, T)$ and T lying in a bounded set.

We further estimate

$$\begin{aligned}
\psi(T_1) &= \|h\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} \\
&\leq (c+1)e^{cT_1} \|g\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} \\
&\leq (c+1)e^{cT_1} (\|w\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} + \|e^{-tA}u(\tau)\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)}) \\
&\leq C_7(c+1)e^{cT_1} (\|w\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} \\
&\quad + \|e^{-tA}u(\tau)\|_{L^q(0,T_1;X_0^s)} + 2\|e^{-tA}Au(\tau)\|_{L^q(0,T_1;X_0^s)}) \\
&\leq C_7(c+1)e^{cT_1} (\|w\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} \\
(5.76) \quad &\quad + \|e^{-tA}\|_{L^q(0,T_1;\mathcal{L}(X_0^s))} (\|u(\tau)\|_{X_0^s} + 2\|Au(\tau)\|_{X_0^s})),
\end{aligned}$$

for some constant $C_7 > 0$ independent of τ , T and T_1 . Therefore, by (5.67), (5.69), (5.74) and Proposition 2.8 we find that $\psi(T_1) \rightarrow 0$ as $T_1 \rightarrow 0$ uniformly in $\tau \in [\delta, T)$ and T lying in a bounded set.

Fix $r_0 > 0$ and assume that $r \in (0, r_0]$. Consider the following set

$$\begin{aligned}
\Sigma_{r,T_1} &= \left\{ a \in W^{1,q}(0, T_1; X_0^s) \cap L^q(0, T_1; X_2^s) \mid \right. \\
&\quad \left. a(0) = u(\tau) \quad \text{and} \quad \|a - h\|_{W^{1,q}(0,T_1;X_0^s) \cap L^q(0,T_1;X_2^s)} \leq r \right\}.
\end{aligned}$$

Further, let the map $\gamma_{r,T_1} : \Sigma_{r,T_1} \mapsto \Sigma_{r,T_1}$ defined by $\gamma_{r,T_1}(a) = b$, where b is the unique in $W^{1,q}(0, T_1; X_0^s) \cap L^q(0, T_1; X_2^s)$ solution of

$$\begin{aligned}
b'(t) + (\underline{\Delta}_s + 1)^2 b(t) &= F(t, a(t)), \quad t \in (0, T_1), \\
b(0) &= u(\tau).
\end{aligned}$$

Note that the existence of b follows since $F(t, a(t)) \in L^q(0, T_1; X_0^s)$ due to (2.5), Lemma 3.2 and (5.62). The proof of Theorem 2.11 is based on the Banach fixed point theorem applied to γ_{r,T_1} . Namely, it is shown that for sufficiently small r and T_1 , γ_{r,T_1} becomes a well defined map which is also a contraction. The constant bounds that determine the above two properties of γ_{r,T_1} are estimated in [3, (2.11)] and [3, (2.14)] and are given by [3, (2.13)] and [3, (2.17)] respectively. Note that in our case [3, (2.13)] and [3, (2.17)] are simplified due to the semilinearity. In general, they are determined by the following five parameters, namely:

- (i) The maximal L^q -regularity bound M from [3, Corollary 2.3].
- (ii) The Lipschitz bound L in [3, (2.9)-(2.10)].
- (iii) The initial data $u(0)$ in $(X_2^s, X_0^s)_{\frac{1}{q}, q}$.
- (iv) The decay of $\phi(z)$ as $z \rightarrow 0$.
- (v) The decay of $\psi(z)$ as $z \rightarrow 0$.

Take $u(\tau)$ as initial data for (1.1)-(1.2) with τ sufficiently close to T . Denote by T_{\max} the supremum of all possible T and assume that $T_{\max} < \infty$. The bound M in our case is determined by (2.29) applied to the operator $(\underline{\Delta}_s + 1)^2$ as well as by the norm of the embedding (2.5), and hence stays uniformly bounded in τ , T and T_1 . The constant $\|u(\tau)\|_{(X_2^s, X_0^s)_{\frac{1}{q}, q}}$ also stays uniformly bounded in τ , T and T_1 due to (5.67). Furthermore, by (5.55), the Lipschitz constant L is determined by the $(X_2^s, X_0^s)_{\frac{1}{q}, q}$ -norm of the solution u , and hence it is uniformly bounded in τ , T and T_1 as well. Next, the right hand side of [3, (2.8)] becomes arbitrary small uniformly in τ , T and T_1 , by taking r_0 and T_2 sufficiently small. Finally, both $\phi(T_1)$ and $\psi(T_1)$ become arbitrary small uniformly in τ and T , by taking the new time step T_1 sufficiently small. Therefore, by taking T sufficiently close to T_{\max} , we can repeat the Clément-Li Banach fixed point step of Theorem 5.1 in order to obtain a

$$W^{1,q}(T_{\max} - \rho, T_{\max} + \rho; X_0^s) \cap L^q(T_{\max} - \rho, T_{\max} + \rho; X_2^s)$$

solution of (1.1)-(1.2), for some $\rho \in (0, T_{\max})$. Such a solution is also unique in

$$W^{1,q}(T_{\max} - \rho, T_{\max} - \eta; X_0^s) \cap L^q(T_{\max} - \rho, T_{\max} - \eta; X_2^s)$$

for each $\eta \in (0, \rho)$, and therefore it coincides with u in the above space, which contradicts the maximality of T_{\max} .

Now if T in Theorem 5.1 can be taken arbitrary large, then (5.63) holds due to (5.59). \square

6. THE SWIFT-HOHENBERG EQUATION ON CLOSED MANIFOLDS

In this section we study the Swift-Hohenberg equation on a closed manifold by employing the singular analysis results described in the previous sections. More precisely, recall that for the case of a conic manifold, in addition to the existence and uniqueness of the solution of the Swift-Hohenberg equation, we obtained information about the asymptotic behavior of the solution close to the singularity. Further, we determined the dependence of this asymptotic behavior in terms of the geometry of the cross section of the cone. Therefore, by taking geodesic polar coordinates on a closed manifold and, under local geometric assumptions, by regarding the space as a manifold with conical singularity, we can derive a relation between the local distribution described by the solution and the local geometry.

Geodesic polar coordinates. Let \mathcal{M} be a closed (i.e. compact without boundary) connected $(n+1)$ -dimensional Riemannian manifold, endowed with a Riemannian metric \mathfrak{f} . Take an arbitrary point $o \in \mathcal{M}$ and denote by \mathbb{S}^n the unit sphere $\{z \in \mathbb{R}^n \mid |z| = 1\}$. If $z \in \mathcal{M} \setminus \{o\}$, let $x = d(o, z)$ be the geodesic distance between o and z , where d is the metric distance induced by \mathfrak{f} . Then, there exists an $r > 0$ such that $(x, y) \in [0, r) \times \mathbb{S}^n$ form local coordinates around o and the metric in these coordinates admits the structure

$$\mathfrak{f} = dx^2 + x^2 \mathfrak{f}_{\mathbb{S}^n}(x),$$

where $x \mapsto \mathfrak{f}_{\mathbb{S}^n}(x)$ is a smooth family of Riemannian metrics on \mathbb{S}^n (see e.g. [20, Lemma 5.5.7]). According to the above choice and under certain local geometric assumption, we regard $\mathbb{M} = (\mathcal{M}, \mathfrak{f})$ as a conic manifold with one isolated warped conical singularity at the pole o . When the Laplacian $\Delta_{\mathfrak{f}}$ on \mathcal{M} induced by \mathfrak{f} is restricted to the collar neighborhood $(0, r) \times \mathbb{S}^n$, according to (4.35), it obtains the form

$$\Delta = x^{-2} \left((x\partial_x)^2 + (n-1 + \frac{x\partial_x(\det(\mathfrak{f}_{\mathbb{S}^n}(x)))}{2\det(\mathfrak{f}_{\mathbb{S}^n}(x))})(x\partial_x) + \Delta_{\mathfrak{f}_{\mathbb{S}^n}(x)} \right),$$

where $\Delta_{\mathfrak{f}_{\mathbb{S}^n}(x)}$ is the Laplacian on \mathbb{S}^n induced by the metric $\mathfrak{f}_{\mathbb{S}^n}(x)$.

Under certain assumption on the local geometry, the existence, uniqueness and regularity results obtained in the previous section are applied to the above setting and imply the following result concerning the problem (1.1)-(1.2) on $(\mathcal{M}, \mathfrak{f})$.

Theorem 6.1. Let $s \geq 0$ and p, q, γ be chosen as in (5.51)-(5.52). Let $o \in \mathbb{M}$ be such that the family of metrics $\mathfrak{f}_{\mathbb{S}^n}(x)$ is smooth up to $x = 0$ and does not degenerate up to $x = 0$. Then, for any

$$u_0 \in (\mathcal{D}(\underline{\Delta}_s^2), \mathcal{H}_p^{s, \gamma}(\mathbb{M}))_{\frac{1}{q}, q}$$

there exists a $T > 0$ and a unique

$$u \in W^{1, q}(0, T; \mathcal{H}_p^{s, \gamma}(\mathbb{M})) \cap L^q(0, T; \mathcal{D}(\underline{\Delta}_s^2))$$

solving the problem (1.1)-(1.2) on \mathbb{M} , where the bi-Laplacian domain is described in (4.48). Furthermore, for each $\delta \in (0, T)$ we have that

$$u \in W^{1, \infty}(\delta, T; \mathcal{H}_p^{s, \gamma}(\mathbb{M})) \cap L^\infty(\delta, T; \mathcal{D}(\underline{\Delta}_s^2))$$

with $u'(t) \in \mathcal{H}_p^{s, \gamma}(\mathbb{M})$ and $u(t) \in \mathcal{D}(\underline{\Delta}_s^2)$ for each $t \in [\delta, T]$.

Therefore, for all $\varepsilon > 0$ sufficiently small we have that

$$u \in C([0, T]; \mathcal{H}_p^{s+2-\delta_{q, \varepsilon}, \gamma+2-\delta_{q, \varepsilon}}(\mathbb{M}) \oplus \mathbb{C}) \hookrightarrow C([0, T]; C(\mathbb{M})),$$

and there exists some $c_0 \in C([0, T]; \mathbb{C})$ such that

$$|u(x, y, t) - c_0(t)| \leq Cx^{\gamma - \frac{n-3}{2} - \delta_{q, \varepsilon}}, \quad x \in [0, r), y \in \mathbb{S}^n, t \in [0, T],$$

for some constant $C > 0$ depending on the data $\{o, u_0, s, p, q, \gamma, T, \delta_{q,\varepsilon}, r\}$, where

$$\delta_{q,\varepsilon} = \begin{cases} \frac{2}{q} + \varepsilon & \text{if } q \leq 2 \\ 0 & \text{if } q > 2 \end{cases}.$$

Hence, if we further assume that the greatest non-zero eigenvalue λ_1 of $\Delta_{\mathbb{f}_{\mathbb{S}^n}(0)}$ satisfies $-\lambda_1 \geq 2(n+1)$, then the weight γ is chosen in $(\frac{n-3}{2} + \frac{2}{q}, \frac{n+1}{2})$ depending on the initial data u_0 .

Proof. Existence, uniqueness and regularity of the solution follow by Theorem 5.1. The asymptotic behavior close to the pole o follows by the embedding (2.5), Lemma 3.2, Corollary 5.2 and Proposition 5.3. Finally, if $-\lambda_1 \geq 2(n+1)$, then in (5.52) we have that

$$\frac{n+1}{2} \leq -1 + \sqrt{\left(\frac{n-1}{2}\right)^2 - \lambda_1}.$$

□

Remark 6.2. For the dependence of the lower eigenvalue λ_1 in terms of the local geometry we refer to [14] and [15]. According to Theorem 5.4, if in Theorem 6.1 we assume that the $L^q(0, T; \mathcal{H}_p^{s,\gamma}(\mathbb{M}))$ -norm of the right hand side of (1.1) as a function of T is bounded by a positive continuous function on \mathbb{R} , then we can proceed to long time solution.

In order to emphasize the relation between the local geometry and the evolution explained in Theorem 6.1, we study the case of constant initial data, which is not a stationary solution in general. We impose some further local geometric assumption and examine the effect on the local radial inhomogeneity of the distribution described by (1.1)-(1.2).

Example 6.3. Let $o \in \mathbb{M}$ be such that the family of metrics $\mathbb{f}_{\mathbb{S}^n}(x)$ is smooth up to $x = 0$ and does not degenerate up to $x = 0$ and assume that the greatest non-zero eigenvalue λ_1 of $\Delta_{\mathbb{f}_{\mathbb{S}^n}(0)}$ satisfies $-\lambda_1 \geq 2(n+1)$. Choose the initial data $u_0 = \text{constant on } \mathcal{M}$. Then, for any $\nu \geq 0$, $\varepsilon \in (0, 1)$ and $q \in (\frac{2}{\varepsilon}, \infty)$ there exists a $T > 0$ and a unique in

$$W^{1,q}(0, T; \mathcal{H}_{\frac{n+1}{2-\varepsilon}}^{\nu, \frac{n+1}{2}-\varepsilon}(\mathbb{M})) \cap L^q(0, T; \mathcal{H}_{\frac{n+1}{2-\varepsilon}}^{\nu+4, \frac{n+9}{2}-\varepsilon}(\mathbb{M}) \oplus \mathbb{C})$$

solution

$$u \in C([0, T]; \mathcal{H}_{\frac{n+1}{2-\varepsilon}}^{\nu+2, \frac{n+5}{2}-\varepsilon}(\mathbb{M}) \oplus \mathbb{C}) \hookrightarrow C([0, T]; C(\mathbb{M}))$$

of the problem (1.1)-(1.2). Therefore, there exists some $c_0 \in C([0, T]; \mathbb{C})$ such that

$$|u(x, y, t) - c_0(t)| \leq Cx^{2-\varepsilon}, \quad x \in [0, r], y \in \mathbb{S}^n, t \in [0, T],$$

for some constant $C > 0$ depending on the data $\{o, u_0, \nu, \varepsilon, q, T, r\}$.

Proof. The result follows from Theorem 6.1 by choosing the weight γ according to the additional geometric assumption and the specific initial data. More precisely, we can choose s arbitrary large, $p = \frac{n+1}{2-\varepsilon}$, $q > \frac{2}{\varepsilon}$ arbitrary and $\gamma = \frac{n+1}{2} - \varepsilon$, for any $\varepsilon \in (0, 1)$. □

Remark 6.4. If in Example 6.3 we further assume that for some $\nu \geq 0$, $\varepsilon \in (0, 1)$ and $q \in (\frac{2}{\varepsilon}, \infty)$ the $L^q(0, T; \mathcal{H}_{\frac{n+1}{2-\varepsilon}}^{\nu, \frac{n+1}{2}-\varepsilon}(\mathbb{M}))$ -norm of the right hand side of (1.1) as a function of T is bounded by a positive continuous function on \mathbb{R} , then, by Theorem 5.4, T can be taken arbitrary large.

Remark 6.5. We can follow similar approach to different problems which can be already solved on manifolds with warped conical singularities. For this purpose we mention the case of the Cahn-Hilliard equation treated in [25, Section 5]. Additionally, the purely quasilinear problem of the porous medium equation was considered in [26] on manifolds with straight cones. The results can be extended to the case of manifolds with warped conical tips by using Theorem 4.1.

REFERENCES

- [1] H. Amann. *Linear and quasilinear parabolic problems*. Monographs in Mathematics Vol. **89**, Birkhäuser Verlag (1995).
- [2] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Monographs in Mathematics Vol. **96**, Birkhäuser Verlag (2001).
- [3] P. Clément and S. Li, *Abstract parabolic quasilinear equations and application to a groundwater flow problem*. Adv. Math. Sci. Appl. **3**, Special Issue, 17–32 (1993/94).
- [4] P. Clément and J. Prüss. *An operator-valued transference principle and maximal regularity on vector-valued L_p -spaces*. In: G. Lumer and L. Weis (eds.), Proc. of the 6th. International Conference on Evolution Equations. Marcel Dekker (2001).
- [5] M. Cross and P. Hohenberg. *Pattern formation outside of equilibrium*. Rev. Mod. Phys. **65**, 851–1123 (1993).
- [6] G. Da Prato and P. Grisvard. *Sommes d’opérateurs linéaires et équations différentielles opérationnelles*. J. Math. Pures Appl. **54** (9), no. 3, 305–387 (1975).
- [7] G. Dore and A. Venni. On the closedness of the sum of two closed operators. Math. Z. **196**, 189–201 (1987).
- [8] J. Gil, T. Krainer and G. Mendoza. *Resolvents of elliptic cone operators*. J. Funct. Anal. **241**, 1–55 (2006).
- [9] M. Haase. *The functional calculus for sectorial operators*. Operator theory: Advances and applications. Vol. **169**, Birkhäuser Verlag (2006).
- [10] R. Haller-Dintelmann and M. Hieber. *H^∞ -calculus for products of non-commuting operators*. Math. Z. **251**, 85–100 (2005).
- [11] M. Kaip and J. Saal. *The permanence of R -boundedness and property (α) under interpolation and applications to parabolic systems*, J. Math. Sci. Univ. Tokyo **19**, no. 3, 359–407 (2012).
- [12] N. Kalton and L. Weis. *The H^∞ -calculus and sums of closed operators*. Math. Ann. **321**, no. 2, 319–345 (2001).
- [13] P. C. Kunstmann and L. Weis. *Perturbation theorems for maximal L_p -regularity*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **30** (4), no. 2, 415–435 (2001).
- [14] P. Li and S. T. Yau. *Estimates of eigenvalues of a compact Riemannian manifold*. AMS Proc. Symp. Pure Math., **36**, 205–239 (1980).
- [15] A. Lichnerowicz. *Géométrie des groupes de transformations*. Dunod (1958).
- [16] M. Lesch. *Operators of Fuchs type, conical singularities, and asymptotic methods*. Teubner Verlag, Stuttgart (1997).
- [17] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser Verlag (2012).
- [18] A. Mielke. *Instability and Stability of Rolls in the Swift-Hohenberg Equation*. Commun. Math. Phys. **189**, 829–853 (1997).
- [19] L. A. Peletier and W. C. Troy. *Spatial Patterns, Higher Order Models in Physics and Mechanics*. Birkhäuser Verlag (2001).
- [20] P. Petersen. *Riemannian Geometry*. Graduate Texts in Mathematics **171**, Springer Verlag (2016).
- [21] Y. Pomeau and P. Manneville. *Wavelength selection in cellular flows*. Phys. Lett. A, **75**, 296–298 (1980).
- [22] J. Prüss and R. Schnaubelt. *Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time*. J. of Math. Anal. and Appl. **256**, 405–430 (2001).
- [23] J. Prüss and G. Simonett. *H^∞ -calculus for the sum of non-commuting operators*. Transactions Amer. Math. Soc. **359**, 3549–3565 (2007).
- [24] N. Roidos. *On the inverse of the sum of two sectorial operators*. J. Funct. Anal. **265** (2) 208–222 (2013).
- [25] N. Roidos and E. Schrohe. *Bounded imaginary powers of cone differential operators on higher order Mellin-Sobolev spaces and applications to the Cahn-Hilliard equation*. J. Differential Equations **257**, 611–637 (2014).
- [26] N. Roidos and E. Schrohe. *Existence and maximal L_p -regularity of solutions for the porous medium equation on manifolds with conical singularities*. Comm. Partial Differential Equations, Vol. **41** (9), 1441–1471 (2016).
- [27] E. Schrohe and J. Seiler. *The resolvent of closed extensions of cone differential operators*. Can. J. Math. **57**, 771–811 (2005).
- [28] J. Swift and P. Hohenberg. *Hydrodynamic fluctuations at the convective instability*. Phys. Rev. A. **15**, 319–328 (1977).
- [29] M. Tlidi, M. Georgiou and P. Mandel. *Transverse patterns in nascent optical bistability*. Phys. Rev. A, **48-2**, 4506–4609 (1993).
- [30] H. Uecker. *Diffusive stability of rolls in the two-dimensional real and complex Swift-Hohenberg equation*. Comm. Partial Differential Equations **24**, no. 11-12, 2109–2146 (1999).
- [31] L. Weis. *Operator-valued Fourier multiplier theorems and maximal L_p -regularity*. Math. Ann. **319**, 735–758 (2001).